X. *The Aerodynamics of a Spinning Shell.*

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Contents.

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INTRODUCTION.

THIS paper contains the results, theoretical and experimental, of work undertaken, at the request of the Ordnance Committee, by the authors as Technical Officers of the Munitions Inventions Department. Permission to publish such parts as appear to be of general scientific interest has now been granted by the Ordnance Committee and the Director of Artillery. The publication of this paper has received their sanction.

The experiments in question were carried out at the firing ground of H.M.S. "Excellent," Portsmouth; the Experimental Department, H.M.S. "Excellent," also provided the 3-inch guns used and the material for the construction of the range. The authors' best thanks are due to the officers of this department, especially Lieut.-Commander R. F. P. MATON, O.B.E., R.N., without whose cordial co-operation these experiments could never have been carried out; also to the other officers of the Munitions Inventions Department who assisted in the heavy work of making and analysing the observations. The aeronautical measurements at low velocities, required for comparison, were made in the wind channels of the National Physical Laboratory, by arrangement with the Director and the Superintendent of the Aeronautical Department, to whom also we wish to express our thanks.

The subject of this paper is the motion of a spinning shell through air at velocities both greater and less than the velocity of sound. We first attempt to describe the motion of the spinning shell, considered as a rigid body, under the effects of gravity and the reaction of the air; this latter is supposed to be known in terms of the position and velocity co-ordinates of the shell, and the state of the air through which it moves. We are thus concerned throughout with the "aerodynamical" problem of the motion of the shell alone, and not with the general "hydrodynamical" problem of the motion of the complete system formed by the shell and air together. The motion of the shell thus described is then compared with the results of experiments, and the more important components of the force system imposed by the air are determined numerically as functions of certain variables such as the velocity of the centre of gravity of the shell. The actual experiments consist of observations of the initial motion of the shell (more particularly the angular motion of its axis of symmetry), over a limited range near the muzzle of the gun. The velocities experimented with range from 40 f.s.* to 2300 f.s., that is from about 0·04 to 2·1 times the velocity of sound. Using the values of the components so determined, the actual motion of the shell can be calculated with equal certainty in the more general cases which are inaccessible to direct and detailed observation.

As stated above, we make no attempt to attack the hydrodynamical problem. Such an attack is probably not yet feasible. By obtaining, however, an accurate descriptive knowledge of the force system imposed by the air, and the allied system

* This velocity was obtained in a wind channel, using a current of air and stationary shell. The lowest velocity used in actual firing experiments was 880 f.s.

of pressure distribution over the surface of the shell, material is provided on which a successful attack on the hydrodynamical problem may some day be based. A first contribution to a knowledge of the force system is made by the present paper. It is hoped to make a similar contribution to a knowledge of the pressure distribution in another place.*

The problem proposed for discussion is of course by no means novel.† In the earlier work which is summarised by CRANZ (*cf.* (4)) the treatment of the equations of motion is often open to criticism, in view of the lack of sufficient justification for the necessary simplifying approximations. The classical theoretical results, such as MAYEVSKI'S equation for the *drift*‡ (see § 4.2, equation (4.204)) have therefore hitherto justly commanded little confidence. The discussion, moreover, is of necessity based on *a priori* assumptions as to the nature of the complete force system. Unless the results of these assumptions are brought to the test of detailed experiment, the assumptions themselves must remain unjustified and unjustifiable. It should be stated here that the theory and experiments described in this paper confirm the classical theoretical results. CRANZ'S own experiments (*cf.* (5)) were expressly designed to explode the fallacy that the axis of the shell, in steady motion, *precesses right round* the direction of motion of its centre of gravity. In this they are successful, but they were only carried out at low velocities, and give little in the way of quantitative results. The only real comparison of theory with experiment, which has hitherto been made, is the comparison of the observed and calculated§ values of the drift. But the observed drift is the integrated result of the disturbing forces over a considerable arc of the trajectory, and moreover, can only be disentangled with difficulty from the effect of any cross wind that may be blowing. The observed drift does not therefore serve to determine the force system with any success, though it may be used to check the values of the components otherwise determined (§ 4.21).

It may, therefore, be stated in general terms that, up to the present, there is no

* "The Pressure Distribution on the Head of a Shell Moving at High Velocities," 'Roy. Soc. Proc.,' A, Vol. XCVII., p. 202.

† See for example:—

P. CHARBONNIER. (1) 'Traité de Balistique Extérieure,' ed. 2, Bk. V., Ch. IV. (2) 'Balistique Extérieure Rationnelle,' vol. II., Ch. IX.

C. CRANZ. (3) 'Lehrbuch der Ballistik; Aeussere Ballistik,' 1917, Ch. X. (4) 'Encyklopädie der Mathematischen Wissenschaften,' vol. IV., Part II., p. 185, Art. 18, "Ballistik." (5) 'Zeitschrift für Mathematik und Physik,' vol. XLIII., pp. 133, 169.

J. PRESCOTT. (6) 'Phil. Mag.,' Ser. 6, vol. XXXIV., p. 332.

Further references to previous authors will be found in (4), and the best account of CRANZ'S own work in (5).

‡ The lateral departure of the projectile from the vertical plane containing the initial tangent to the path of the centre of gravity of the shell.

§ Actually, also, the important term in the calculated drift depends only on the ratio of two components of the force system, and not on their absolute values.

knowledge of the force system acting on any shell at high velocities, except when the shell is moving "nose on," *i.e.*, when its axis of symmetry and the direction of motion of its centre of gravity coincide.*

CRANZ (*cf.* (5)) and CHARBONNIER (*cf.* (2)) make little progress in the treatment of the general equations of motion. PRESCOTT (*cf.* (6)) makes an appreciable advance in the reductions of the general equations of motion to a tractable form, which is not too restricted in application, and gives an exact solution of his reduced equations in the simple case in which all the components of the impressed force system vary as the square of the velocity of the shell. We understand, also, that the problem of the initial motion of the shell has been recently treated by M. ESCLANGON and M. GARNIER, of the French Artillerie de Marine, with results that are closely analogous to ours, but we have not seen their work.

We therefore propose, in this paper, to give in Part III. a detailed account of the complete equations of motion of a spinning shell, moving through air, and to justify as far as possible the reduction of these equations to various useful approximate forms, some of which are classical. To do this, it is of course necessary to start from certain *a priori* assumptions as to the nature of the complete force system. These assumptions, which are far less restrictive than any that have hitherto been used, are carefully analysed when they are introduced. We then, in Part IV., submit the theoretical results so obtained to the test of the experiment described in Part II.; we are thus able to justify to some extent our *a priori* assumptions, and to obtain numerical results of some precision as to the more important components of the force system acting on the shell, in the general case. These numerical results, with a general description of the actual motion of a shell, will be found in Part I.

We have seen that the information to be obtained by comparison of the observed and calculated values of the drift is of very limited value. Two alternative methods are available, both of which are employed in this paper:—

(1) The complete force system on a model shell at rest in a uniform current of air may be determined by observations in a wind channel.†

(2) Certain components of the force system on a shell moving at high velocity may be deduced from the measurements of its oscillations just after leaving the muzzle.

The highest velocity obtainable at present by the first method is 80 f.s., but by means of the "square law" (see § 1.01) the results may be extended to velocities as

* In this case the force system has only one component of practical importance, namely, the resistance of the air, acting in the opposite direction to the relative motion of air and shell. This force component is here called *the drag*, in conformity with aerodynamical usage. The numerical values of the drag are known with fair accuracy for certain external shapes of shell and ordinary atmospheric conditions.

† For a full description of the construction of the wind channels at the National Physical Laboratory, and their use in measuring forces on model aircraft, see COWLEY and LEVY, "Aeronautics in Theory and Experiment."

great as 700 f.s. For higher velocities it is necessary to fall back on the second method which is the principal subject of this paper.

For this purpose the shell is fired horizontally through a series of cards such as are used for measuring the jump* of the gun on firing. From the shape of the holes in the cards the actual motion of the axis of the shell can be reconstructed. Initial disturbances at the muzzle give rise to angular oscillations of the shell of sufficient amplitude for accurate measurement. These oscillations are very similar to those of the axis of a spinning top under gravity. If, as a first approximation, we regard the centre of gravity of the shell as constrained to move uniformly in a straight line over the range containing the cards, and ignore frictional damping forces in both cases, then the angular motion of the axis of the top and the axis of the shell are identical, provided that (1) the top and shell have the same axial spin and axial moment of inertia; (2) the transverse moment of inertia of the top about its point of support is equal to the transverse moment of inertia of the shell about its centre of gravity; and (3) the moment of gravity about the point of the top is equal to the moment of the force system on the shell about its centre of gravity.

In this approximate case the formal solution of the two problems is identical. As is explained in § 1.3, from the periods of the oscillations of the axis of the top or shell, we can deduce the moment of the disturbing couple and *vice versa*. In the same way the nature of the decay of the oscillations can be used to determine the damping forces.

In conclusion, we feel that a word of apology may be needed for the length of the introductory part of this paper. We do not here emphasise the applications to practical gunnery of the results obtained, but these are of some importance. We have, therefore, thought it desirable that the results should be presented in such a form as to be available to those who are concerned with the practical results, but who are not prepared to follow in detail the arguments of Parts III. and IV. At the same time it has been necessary to avoid statements which, without explanations, might convey little meaning to those who have not been technically concerned with ballistics and aerodynamics. It does not appear possible to achieve these objects except at the expense of a somewhat lengthy Introduction and Part I.

PART I.—A GENERAL DESCRIPTION OF THE MOTION OF A SPINNING SHELL AND THE PRINCIPAL EXPERIMENTAL RESULTS.

§ 1.0. *The Classical Theory of the Plane Trajectory.*

According to the classical theory, a shell is supposed to move in a resisting medium like a particle on which the only forces acting are gravity, and a resistance tangential to its path, depending only on the velocity of the particle and the state of the

* The angle between the axis of the bore before firing and the initial tangent to the path of the centre of gravity of the shell.

undisturbed medium. In such circumstances the path of the particle lies in a vertical plane and is called *the plane trajectory.** This theory would be exact for a shell if the axis of the shell always pointed along the tangent to the path of its centre of gravity. The total reaction between the air and the shell would then, as required, take the form of a single force, called the drag, acting by symmetry tangentially to the path of the centre of gravity, and depending only on the velocity and shape of the shell and the state of the medium. The equations of motion resulting in this simple case are insoluble in finite terms for the actual law of resistance of the air; in practice they are capable of rapid numerical solution to any desired degree of accuracy, by a variety of methods of step-by-step integration, when the drag has been specified with corresponding accuracy.

In order to specify the drag completely it is necessary to consider with some care what are the variables on which the drag for a given shell can depend to an appreciable extent. This question is, as yet, by no means settled, and a few of the more important considerations are summarised in § 1.01. This fact does not concern us here to a very serious extent; an incomplete specification of the variables on which the drag (or, in the general case, the complete force system) depends will only invalidate the results of observation when an attempt is made to apply them to widely different conditions of the state of the resisting medium, or of the motion of the shell. The validity is unaffected when the experimental conditions are approximately repeated. It may be assumed that, in this case of symmetry, a fairly adequate expression for the drag is given by the equation

$$(1.001) \qquad \mathrm{R} = \rho v^2 r^2 f_{\mathrm{R}}(v/a),$$

where R is the total drag, ρ the density of the air (or other medium), r the radius of the shell, v the velocity of the shell, and a the velocity of sound in the undisturbed medium; all these quantities, of course, are to be measured in a consistent set of units. In the numerical work in this paper the foot, pound, second system will be used.

Since $\rho v^2 r^2$ has the dimensions of a force, the function f_{R} is a numerical coefficient, independent of the system of units chosen, called the *drag coefficient.* Existing determinations of this coefficient as a function of v/a are very inadequate from a scientific point of view; satisfactory ones could now be made. We shall not be concerned here with the determination of this coefficient, whose value we shall only require roughly in the analysis of our experiments. We may therefore regard f_{R} as known for all values of the argument from 0 to 3, for shells of the particular external shapes which we use, moving through dry (or not too nearly saturated) air, whose temperature is not too widely different from 0° C.

* From the point of view of this paper, we regard the whole theory of the plane trajectory as "classical," though its adequate treatment was only evolved during the last years of the war.

1.01. *The Functional Form of the Drag Coefficient.*—A careful consideration of the possible forms of the function f_R, from the points of view of the kinetic theory of gases and the theory of dimensions, suggests that γ, l/r, and σ/r should be possible arguments of f_R, besides v/a. Here γ is the ratio of the specific heats of the gas, l is the mean free path, and σ the effective diameter of its molecules. We may, if desired, replace l/r by the more usual viscosity argument vr/ν, where ν is the kinematical coefficient of viscosity. Wind channel work on aerofoil and airscrew models shows that the argument vr/ν is of great importance at low velocities. Its effects, however, in the case of shell models seem almost to have disappeared by the time a velocity of 40 f.s. (or at any rate 75 f.s.) is reached. RAYLEIGH* obtains formulæ for the pressure on a piston moving in a pipe, which show the kind of way in which γ, as well as v/a, might enter into the expression for f_R. Variations of γ are, however, very small in practice. There is experimental evidence that some argument, other than v/a or γ, has an appreciable effect in practice, and that this argument is probably not the viscosity term in the ordinary sense. It is not possible to pursue the question further here, or to assemble in detail the evidence, which is to be found in various minutes of the Ordnance Committee.

So long as the stream lines of the flow remain unaltered by a change of velocity, the motion remains dynamically similar, the drag varies as v^2, and the coefficient f_R must be a constant. The drag is then said to obey *the square law*. Experiments with air screws, of high peripheral speed, appear to show that, up to values of v/a as great as $0\cdot 7$, there is no serious departure from the square law once a certain minimum velocity is exceeded, above which the ordinary viscosity effects become unimportant; this appears, from all the evidence, to be the case also for shells, the minimum velocity being of the order of 50 f.s. As velocities of less than 100 f.s. may be ignored in ballistics, it is therefore customary to assume that the drag obeys the square law exactly for all velocities less than about $0\cdot 7a$. For all such velocities the stream lines of the flow will remain nearly unaltered and the motion will be dynamically similar.

Above this velocity ($0\cdot 7a$.) the effects of the compressibility of the air become rapidly of great importance, and the whole nature of the air-flow changes as a, the velocity of sound, is reached and exceeded. These effects are represented by the variation of f_R as a function of v/a. A good typical curve showing this variation is given by CRANZ.† Another example will be found in fig. 4.

We have so far ignored the fact that the shell is actually spinning about its axis of symmetry. There is no evidence to show that the drag, in the case of symmetry, is appreciably affected by the spin, and its neglect is probably justified.

A more important question is the legitimacy of assuming, as we have tacitly done in (1.001), that the drag does not depend appreciably on the acceleration of the shell. With regard to the acceleration at low velocities, it is known that the effect of the air is to increase the virtual mass of any body by an amount of the order of the mass of air displaced. This is an increase of the order of 1 part in 2000, and is entirely negligible. At higher velocities, and on the general question, direct experimental evidence is unfortunately lacking. It is, however, difficult to see, by theoretical reasoning, how the past history of the shell can have any *large* effect, and there is sufficient general experimental evidence that (1.001) is, on the average, an adequate representation of the drag in the case of symmetry to be certain that the past history is of little importance, except conceivably for a very limited range of velocities, for example, in the neighbourhood of a, the velocity of sound.

§ 1.1. *The Detailed Specification of the Complete Force System.*

The theory discussed in this paper treats the shell as a rigid body which is a solid of revolution, so that its axis of symmetry coincides with a principal axis of inertia.

* "Aerial Plane Waves of Finite Amplitude," 'Scientific Papers,' vol. V., or 'Roy. Soc. Proc.,' A, vol. LXXXIV. See in particular the last section of the paper.

† 'Encyklop. der Math. Wiss.,' vol. IV., Part II., p. 197.

It aims at determining the exact angular motion, as well as the motion of the centre of gravity. It confirms the classical theory of the plane trajectory (in accordance with the results of experiment), by showing that the divergences of the axis of the shell from the tangent to its path are generally small, but it aims, further, at determining the magnitude and effect of these divergences.

In this general case the force system to be specified is more elaborate than in the case of the classical theory. In accordance with aerodynamical usage, we call the angle between the axis of symmetry of the shell and the direction of motion of its centre of gravity *the yaw*, and denote it by δ. When the shell, regarded for the moment as without axial spin, has a yaw δ, and the axis of the shell OA and the direction of motion OP remain in the same relative positions, the force system can by symmetry be represented, as shown in fig. 1, by the following components, specified according to aerodynamical usage.

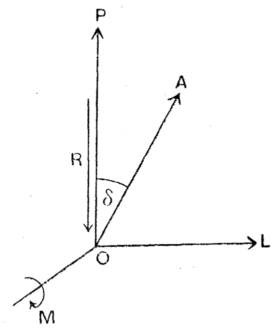


Fig. 1.

(1) *The drag*, R, acting through the centre of gravity O, in the direction of motion OP reversed.

(2) A component L, at right angles to R, called *the cross wind force*, which acts through O in the plane of yaw POA, and is positive when it tends to move O in the direction from P to A.

(3) *A moment* M about O, which acts in the plane of yaw, and is positive when it tends to increase the yaw.

By analogy with § 1.0, we assume the following forms for R, L, and M:—

(1.101) $$R = \rho v^2 r^2 f_R (v/a, \delta),$$

(1.102) $$L = \rho v^2 r^2 \sin \delta f_L (v/a, \delta),$$

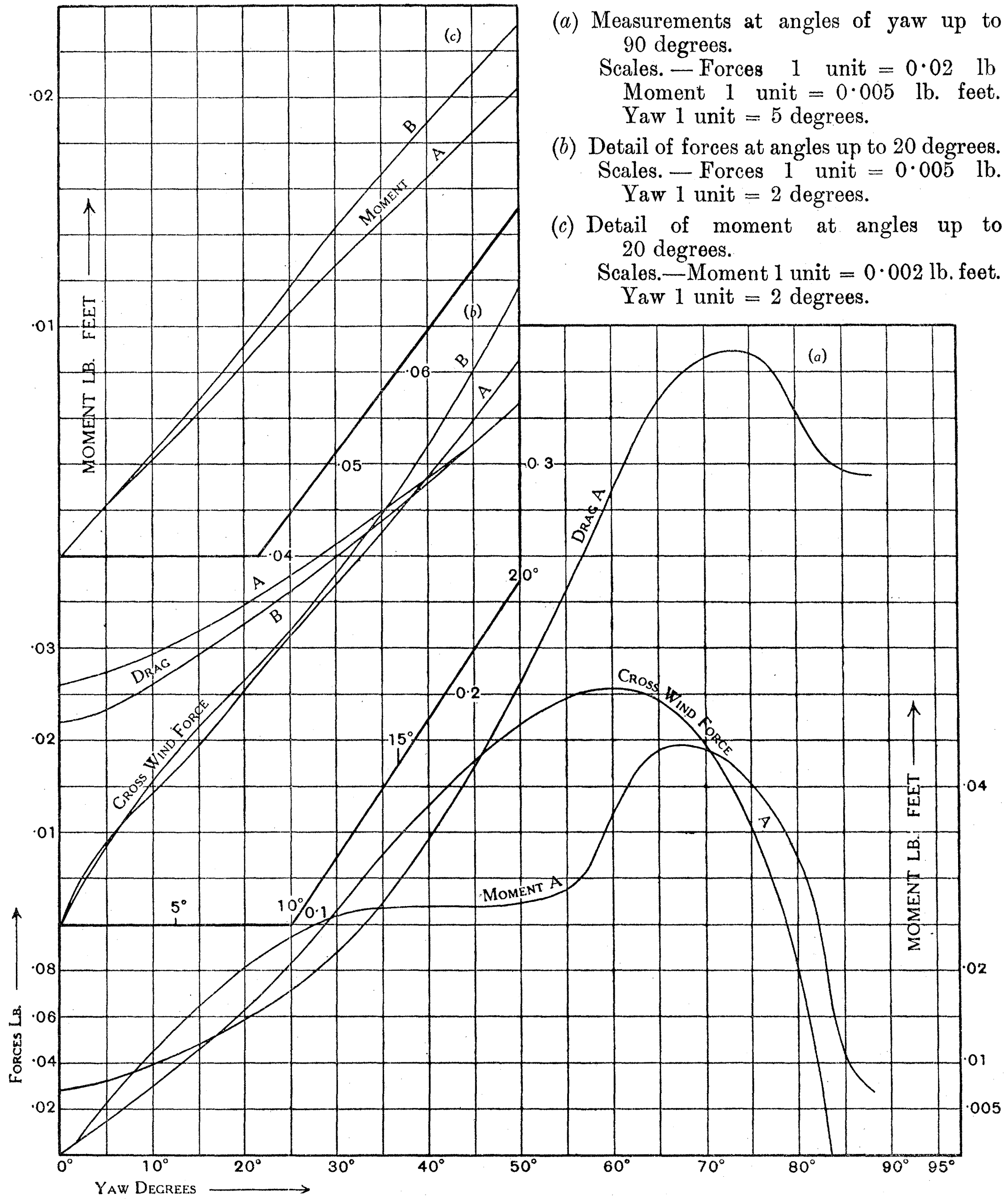
(1.103) $$M = \rho v^2 r^3 \sin \delta f_M (v/a, \delta).$$

These equations are of the most natural forms to make f_R, f_L, and f_M of no physical dimensions. The arguments of § 1.01, by which the form of equation (1.001) was justified to some extent, probably apply with equal force in this more general case. The form chosen is suggested by the aerodynamical treatment of the force system on an aeroplane. Since L and M, by symmetry, vanish with δ, the factor $\sin \delta$ is explicitly included in (1.102) and (1.103), in order that the cross wind force and moment coefficients, f_L and f_M, may have non-zero limits as $\delta \to 0$. We shall use the symbols $f_R (v/a)$, $f_L (v/a)$, $f_M (v/a)$ for $f_R (v/a, 0)$, $\underset{\delta \to 0}{\mathrm{Lt}} f_L (v/a, \delta)$, and $\underset{\delta \to 0}{\mathrm{Lt}} f_M (v/a, \delta)$ respectively, and shall omit the explicit mention of the argument v/a when no confusion can arise by so doing.

In view of the evidence mentioned in § 1.01, we may confidently expect that, for all values of δ, all three coefficients will be nearly independent of v/a in the region $0 \cdot 1 \leq v/a \leq 0 \cdot 7$, and shall, when required, assume their absolute independence of v/a

Fig. 2. Force components on the 3-inch shells A and B, measured in a wind channel at a wind speed of 40 f.s., plotted against angle of yaw.

Shell of form A.—Moment measured about a point 4·85 inches from base.
„ „ B.—Moment measured about a point 4·85 inches from base.



(a) Measurements at angles of yaw up to 90 degrees.
Scales.—Forces 1 unit = 0·02 lb
Moment 1 unit = 0·005 lb. feet.
Yaw 1 unit = 5 degrees.

(b) Detail of forces at angles up to 20 degrees.
Scales.—Forces 1 unit = 0·005 lb.
Yaw 1 unit = 2 degrees.

(c) Detail of moment at angles up to 20 degrees.
Scales.—Moment 1 unit = 0·002 lb. feet.
Yaw 1 unit = 2 degrees.

when $v/a \leq 0\cdot7$. With regard to their dependence on δ we are not here concerned experimentally with f_R. We shall assume for the purpose of analysing our experiments, where only a rough value of f_R is required, that $f_R(v/a, \delta)$ is independent of δ for small values of δ.* For the usual position of the centre of gravity of the shell, f_M at low velocities is remarkably nearly independent of δ for all values less than 10 degrees, and then diminishes as δ increases beyond this value. On the other hand, at low velocities, $f_L(v/a, \delta)$ behaves curiously for small values of δ. The wind-channel value of $f_L(v/a)$ is in consequence uncertain. Typical curves showing f_R, f_L and f_M as functions of δ at low velocities are shown in fig. 2. It is the main purpose of the experimental part of this paper to determine $f_L(v/a)$, and $f_M(v/a)$ as functions of v/a, when $v/a > 0\cdot7$.

1.11. *The Effect of the Angular Motion of the Axis of the Shell.*—In practice the direction of the axis of the shell relative to the direction of motion changes fairly rapidly. By analogy with the treatment of the motion of an aeroplane, we assume, tentatively, that the components of the force system R, L, and M are unaltered by the angular velocity of the axis, but that the effect of the angular motion of the axis of the shell can be represented by the insertion of an additional component, namely, a couple H, called *the yawing moment due to yawing*, which satisfies the equation

(1.111) $$H = \rho v w r^4 f_H(v/a, \ldots),$$

where w is the resultant angular velocity of the axis of the shell. The form of (1.111) is chosen to make f_H of no physical dimensions and is the only one suitable for the purpose. The couple H is assumed to act in such a way as directly to diminish w (see fig. 3). The yawing moment coefficient f_H may be expected to vary considerably with v/a. It may depend appreciably on other arguments such as wr/v and δ. This couple is suggested by, and is analogous to, the more important of the "rotary derivatives" in the theory of the motion of an aeroplane. It appears from considerations of symmetry that no other couple of the "rotary derivative" type need be considered. We shall arrive at rough values of f_H from our experimental results, and to some degree an *a posteriori* justification of our

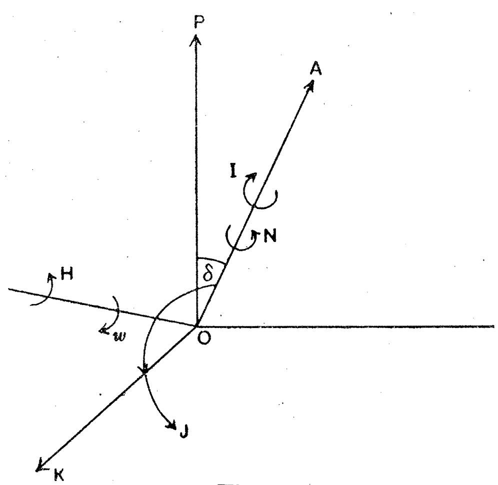


Fig. 3.

* By symmetry $\partial f_R/\partial\delta = 0$, when $\delta = 0$, since f_R has a minimum for $\delta = 0$. It might therefore be expected that, when δ is less than 3 degrees (say), f_R would be nearly independent of δ. This, however, is not the case in wind-channel experiments. The drag at 2 degrees and 3 degrees yaw may be 7 per cent. and 10 per cent. greater, respectively, than the drag at zero yaw. Such evidence as exists indicates that the same increase may occur also at high velocities. An experimental study of the variation of the drag with δ at high velocities would present no insuperable difficulties with modern apparatus.

assumption that L and M are unaffected by the angular velocity of the axis. But the values we obtain are too rough to enable us to study the variations of f_H with any argument.

1.12. *The Effect of the Axial Spin of the Shell.*—We have so far ignored the possible effect of the spin N of the shell about its axis of symmetry. We shall assume that the preceding components of the force system R, L, M and H are not appreciably affected by this spin. This is in accordance with such evidence as exists in the case of zero yaw (§ 1.01). If, moreover, the component M were seriously affected by the spin, the effect would have been detected by the present trial. No such effect was found (see § 4.13), and this fact provides some evidence of the validity of the above assumption, at least as a first approximation.

The spin N will, however, give rise to certain additional components of the complete force system. There will be a couple I which tends to destroy N, and, when the shell is yawed, a sideways force, which need not act through the centre of gravity, analogous to that producing swerve on a golf or tennis ball. This force must, by symmetry, vanish with the yaw. The swerving force must act normal to the plane of yaw, otherwise it would merely have a component which altered R or L (acting in the plane of yaw), and we have assumed that no such component exists. The complete effects of the spin N can therefore be represented by the addition to the force system of the couples I and J and the force K, acting as shown in fig. 3. To procure the correct dimensions we may assume that those components have the forms*

(1.121) $$I = \rho v N r^4 f_I,$$

(1.122) $$J = \rho v N r^4 \sin\delta f_J,$$

(1.123) $$K = \rho v N r^3 \sin\delta f_K.$$

The coefficients f_I, f_J, f_K may depend effectively on a number of variables which we can make no attempt to specify in the present state of our knowledge. These components may be expected to be very small in comparison with L and M; no certain evidence that they exist is given by our experiments.

1.13. *Relations Between the Components of the Force System.*—The various coefficients in the foregoing specifications will all depend on the external shape of the shell; results obtained for one shape cannot be applied to another. For shells of given shape, however, moving in a given manner, the forces R and L are independent of the position of O, the centre of gravity, while the moment M varies with the position of O. If M_1 and M_2 are the values of M corresponding to positions O_1 and O_2 of O, then

(1.131) $$M_1 = M_2 + O_1O_2\,(L\cos\delta + R\sin\delta),$$

where O_1O_2 is positive when O_1 is nearer the base than O_2. Using the relations (1.101) to (1.103), and assuming that the yaw is small, the equation (1.131) reduces to

(1.132) $$f_{M_1} = f_{M_2} + \frac{O_1O_2}{r}(f_L + f_R).$$

This equation is of considerable practical importance, as it enables us to deduce the

* We shall frequently write $\Gamma = I/AN$, where A is the moment of inertia about the axis of symmetry of the shell (see § 1.31).

values of f_L from the values of f_M for two different positions of the centre of gravity. It is found that f_L cannot conveniently be directly observed.

It will be found convenient in the practical use of (1.132) to introduce the force component *normal to the axis of the shell.* If f_N is the corresponding coefficient, it is easily seen that, when the yaw is small,

$$f_N = f_R + f_L \tag{1.133}$$

and that it is f_N that is directly determined by the variations of f_M.

No other relations between the various coefficients are available. Previous to the present experiments, when no definite information existed as to the form of f_L and f_M as functions of v/a, special arbitrary assumptions have been made, in order to carry out calculations of the drift of a shell, or of the twisted curve described by its centre of gravity. The authors have made considerable use of the assumptions that the fractions

$$\frac{f_R(v/a, \delta)}{f_R(v/a, 0)}, \quad \frac{f_L(v/a, \delta)}{f_R(v/a, 0)}, \quad \frac{f_M(v/a, \delta)}{f_R(v/a, 0)}$$

are independent of v/a, and have determined their values by wind-channel observations. CRANZ,* using essentially the same assumptions, has calculated the values of these fractions by an empirical law due to KUMMER. It must be emphasised that the use of any assumption of this type is of very dubious validity, and that, so far as experiments have yet gone, they have not confirmed any such assumptions. When the values of the coefficients f_R, f_M and f_L are required for a shell of any given external shape they can and must be determined by direct experiment.

1.14. In the preceding sections, we have built up, by synthetic arguments, what appears to be the most probable complete force system. It will be seen that in so doing we have actually introduced what can be regarded as a complete system of three forces and three couples referred to three axes at right angles. Owing, however, to the complex nature of the reactions, it appears to us to be essential to construct our force system in this manner, instead of attempting to analyse a complete system of three forces and three couples, and assign each component to its proper causes. In this construction, we have been guided by considerations of symmetry, the theory of dimensions, the analogy with the theory of the aeroplane, and also, of course, by the all-important fact that the results of this construction are in agreement with experiments, so far as these have yet been carried. Of our seven components by far the most important are R, L and M; then, some way behind, H. Our experiments were designed to determine L and M, and if possible to throw some light on the size of H, and in these objects a successful start has been made. As a result, it seems reasonable to expect that the preceding specification of the complete force system will prove to be adequate; but much more work on these and other lines† is still required. With the numerical knowledge already obtained, which is

* 'Zeitschrift für Math. u. Phys.,' vol. XLIII., p. 184.

† For instance, the determination of the couple I that destroys the axial spin and the behaviour of f_R as a function of δ.

given in § 1.2, the motion of a shell, of the shape used in these experiments, can be calculated with some approach to certainty. The general nature of the motion is described in § 1.3.

§ 1.2. *The Numerical Results of the Experiments.*

We now proceed to give the numerical results obtained by analysis of the observations by the methods explained in detail in Part IV.

1.21. *The Values of f_M and f_L.*—The observed values of f_M and f_L are shown plotted against v/a, in fig. 4, for the shell of external form A.* The value of f_M is

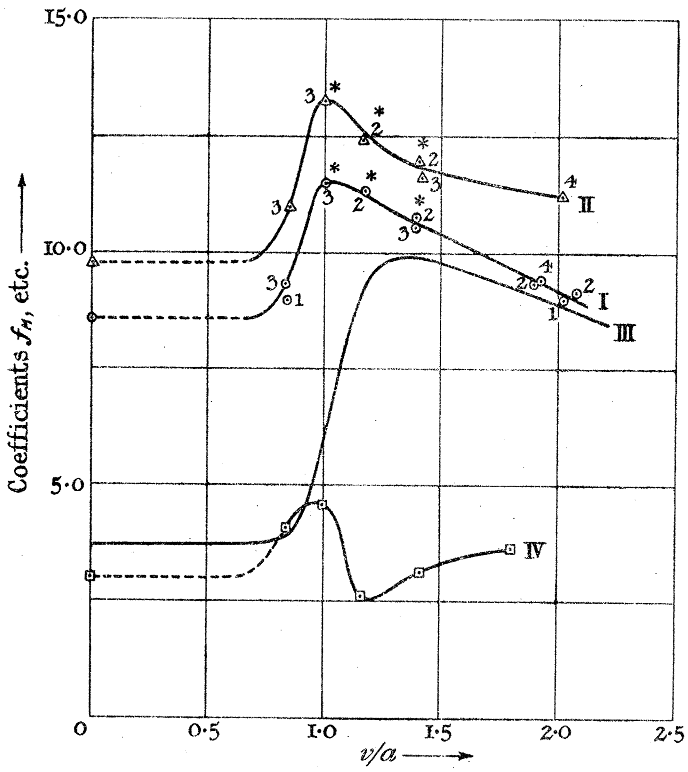


Fig. 4. Shells of form A.

Curve I.—The couple coefficient $f_M(v/a)$ for 3-inch shells, with the centre of gravity 4·73 inches from the base.

Curve II.—The same, with centre of gravity 4·20 inches from the base.

Curve III.—The drag coefficient $f_R(v/a)$ for comparison on ten times the scale.

Curve IV.—The cross-wind force coefficient $f_L(v/a)$.

The plotted points ⊙, △, ⊡ show the observed values. The numbers denote the number of observations whose mean is represented by the plotted point. The stars distinguish those groups fired from the gun rifled one turn in thirty diameters. The others were fired from a gun rifled one turn in forty.

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* See fig. 6. Form A may be specified thus:—Length 3·84 shell diameters. Base cylindrical. Head with an ogive of 2 diameters radius. Centre of gravity 1·577 diameters from base.

given on the assumption that the centre of gravity is 4·73 inches from the base in the 3-inch shells used. The value of f_R (at zero yaw) is also given for comparison. In fig. 5, the corresponding values of f_M and f_R are given for the shells of external form B,* the centre of gravity being supposed to be 4·965 inches from the base in the 3-inch shells used. These values have been corrected as far as possible for the effect of the cards (see § 2.32), and smooth curves have been drawn through the observations. The values of f_M and f_L for shell A, and f_M for shell B, are given in the following table, Table I., for values of v/a varying by 0·1. These values have been read from the smooth curves of the figures. Besides f_L, the value of f_N, the force coefficient

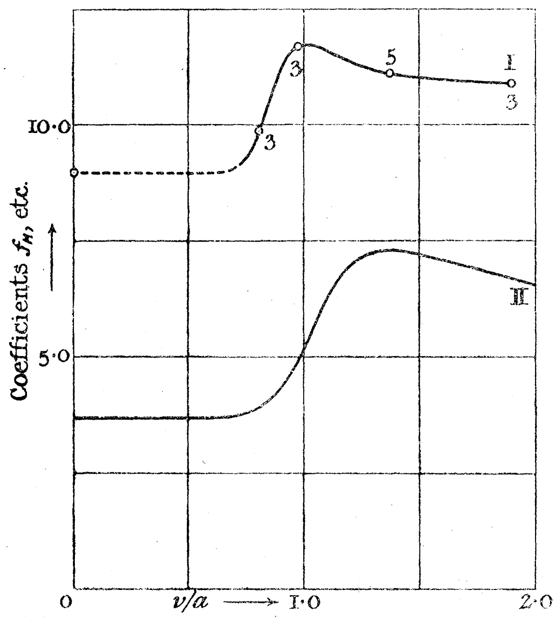


Fig. 5. Shells of external form B.

Curve I.—The moment coefficient $f_M(v/a)$ for 3-inch shells with a centre of gravity 4·965 inches from the base.

Curve II.—The drag coefficient $f_R(v/a)$ shown roughly on ten times the scale.

normal to the shell, is also given. These figures and Table I. represent the main results of the experiment. The values of f_M have a probable error of less than 2 per cent., and the values of f_L of about 10 per cent.

The differences in the various curves for f_R, f_L and f_M are very instructive. They show the complete impossibility of regarding the ratio of f_R/f_M, for example, as constant for large variations of v/a. Unlike f_R, f_M is comparatively unaffected by the velocity of sound. It increases only to about 35 per cent. above its low velocity value, and does not maintain this increase except for a narrow range of velocities near $v/a = 1$. On the other hand f_R increases to two and a-half times its low velocity value and maintains this increase.

* See fig. 6. Form B may be specified thus:—Length 4·34 diameters. Base cylindrical. Head with an ogive of 6 diameters radius. Centre of gravity 1·655 diameters from base.

TABLE I.—Experimental Values of the Couple Coefficient $f_M(v/a)$, the Normal Force and Cross Wind Force Coefficients $f_N(v/a)$ and $f_L(v/a)$, for Shells of Form A, fig. 6; also Values of $f_M(v/a)$ for Shells of Form B, fig. 6.

Determined by firing trials with 3-inch shells.

v/a.	Shell of form A.			Form B.
	$f_M(v/a)$.	$f_N(v/a)$.	$f_L(v/a)$.	$f_M(v/a)$.
Wind channel.	8·57	3·34*	3·0*	8·95
0·7	8·6	—	—	9·05
0·8	9·05	4·3	3·9	9·75
0·9	10·35	—	—	11·15
1·0	11·55	5·2	4·6	11·7
1·1	11·4	—	—	11·6
1·2	11·1	3·5	2·6	11·35
1·3	10·8	—	—	11·15
1·4	10·55	4·1	3·1	11·05
1·5	10·3	—	—	11·0
1·6	10·05	4·3	3·35	11·0
1·7	9·85	—	—	10·95
1·8	9·65	4·5	3·6	10·95
1·9	9·4	—	—	10·90
2·0	9·15	—	—	—

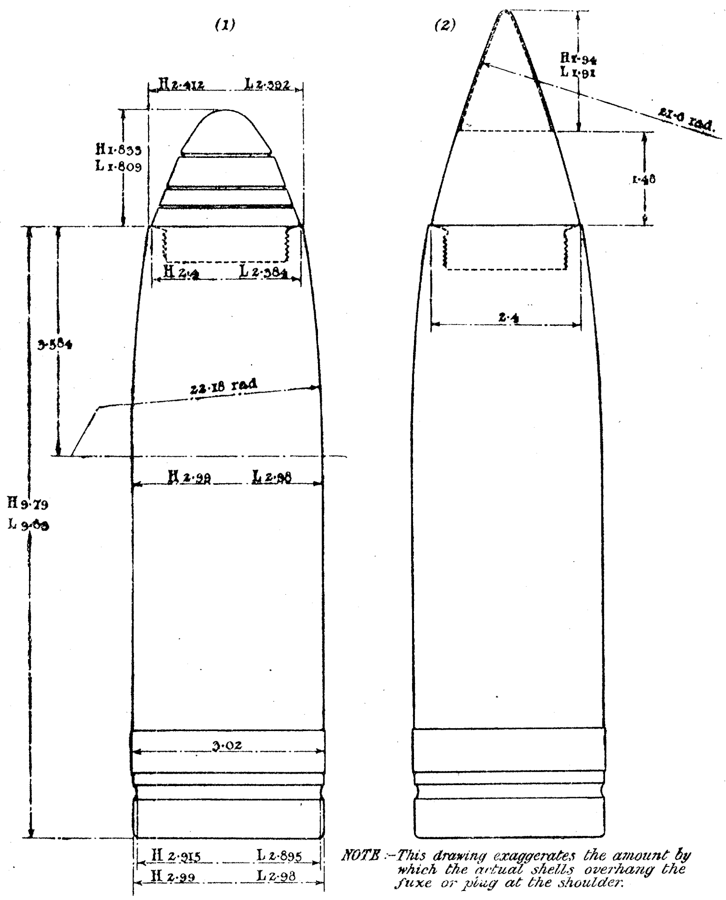


Fig. 6. Showing the external contour of the 3-inch 16-lb. shells, Design H.E. Mark IIB, used in the trial with (1) No. 80 fuze, Mark III; (2) 6 C.R.H. plug, Design 25420.

Note.—The driving band is shown cut off at a diameter of 3·02 inches, its mean diameter after engraving.

* Uncertain.

As already mentioned in § 1.1, the low velocity value of f_L, as determined in the wind channel, is somewhat doubtful.* There appears to be a distinct minimum in this coefficient soon after the velocity of sound, followed by a steady rise. This type of curve is rather unexpected, and confirmation by a repetition of the experiment is very desirable. We may emphasise again that f_L/f_R is by no means constant, and that f_L/f_M (see fig. 15) undergoes considerable variations.

1.22. *The Force Components f_H and Others.*—We have now to exhibit the information obtainable from the damping of the oscillations of the axis. When f_L is known, this information (see § 4.12), provides numerical values for two quantities, one of which is f_H and the other $f_H+\epsilon$, where ϵ depends on the coefficients f_J and f_I and is *a priori* unlikely to be comparable with f_H. The data at our disposal are very rough and could be improved on in future experiments. The present results vary largely in some cases from round to round; the value of ϵ is much larger than its expected value and of the opposite sign. The general features of the damping (see figs. 12, 14) are however clear and qualitatively consistent. We can assert that the following rough values of f_H, given in Table II., are of the right order of magnitude and perhaps not in error by more than 50 per cent. Owing to their roughness they are given for the groups as fired. An attempt has been made to determine f_H in a wind channel at low velocities, the value 22 being obtained.

TABLE II.—Probable Values of f_H, the Coefficient of the Yawing Moment due to Yawing. Groups I., II., III. refer to shells of Form A with various positions of the centre of gravity (see § 2.2). Group IV. refers to Form B.

Group. Muzzle velocity.	f_H.	Group. Muzzle velocity.	f_H.	Group. Muzzle velocity.	f_H.
I. 22–24 1119	80	II. 24 1292	70	III. 1–4 2025	70
I. 25, 26 1326	70	II. 5–7 22, 23† 1587	75	IV. 13–15 1078	55
I. 27, 28 1563	60	II. 1–4 2024	60	IV. 16–18 1547	75
I. 1–4 2167	35	III. 17–19 1119	40	IV. 24–26 2120	80
I. 19–21 2320	30	III. 20, 21 1292	70		
II. 17–19 1119	90	III. 22, 23 1567	60		

* In fig. 4, and subsequently, the low velocity value of f_L is assumed to be $f_L(10°)$ in place of $\mathrm{Lt}_{\delta\to 0}\, f_L(\delta)$, which is uncertain.

† From guns of different riflings, with results in agreement.

An interesting feature of the damping is that, at a velocity of about 900 f.s., the yaw has a distinct tendency *to increase* (*instead of decreasing*) *with the time ; this happens with all four types of shells.* Whether this represents a real phenomenon or is caused by the impacts on the cards (§ 4.5) is not yet clear. It is not physically impossible that f_H may be negative for this velocity. These rounds are ignored here, and further details must be postponed for Part IV.

§ 1.3. *A Description in General Terms of the Angular Motion of the Axis of a Shell.*

The numerical values of f_R, f_L and f_M described in § 1.21, with the addition of the rough values of f_H given in § 1.22 make it possible to determine numerically, by the principles of rigid dynamics, the motion of a shell projected in any manner, provided that the velocity ratio v/a, and the angle of yaw δ, do not pass outside the limits for which the determination is valid. It is necessary to obtain and solve the dynamical equations of motion in terms of the force components before proceeding to the inverse process of deducing the forces from the observed motion of the shell. Before doing so, however, it is convenient to describe in general terms the motion of the shell in various circumstances ; this description is qualitative only, and is inserted for the purpose of illustration : the quantitative results are reserved for Part IV.

1.31. *The Spinning Top Analogy.*—We have already noticed in the Introduction the important analogy between the motion of the axis of a shell and the axis of a spinning top. With the reservations there made, the analogy is complete, so long as f_M can be regarded as independent of δ. The equations of motion of a stable shell, given in § 3.2, are a generalisation of the equations for the small oscillations of a top in the neighbourhood of the vertical. For the general case of stable or unstable motion where the yaw need not be small, some use can be made of the exact equations of motion of the top (§ 3.4).

In particular, the condition for *the stability* of a shell is identical with the condition for a top. The condition that the shell should be in stable equilibrium with its axis parallel to its direction of motion is that

$$A^2N^2 > 4B\mu, \tag{1.311}$$

where A and B are, respectively, the moments of inertia of the shell about longitudinal and transverse axes through the centre of gravity, N is the spin of the shell about its (longitudinal) axis in radians per second, and $\mu \sin \delta$ is equal to M, the moment of the air forces about the centre of gravity. It is therefore convenient to define a new variable s, "the coefficient of stability," by the equation

$$s = A^2N^2/4B\mu. \tag{1.312}$$

When s is greater than unity by a sufficiently large amount, a possible form of angular motion for both shell and top consists of a small oscillation, composed of periodic terms with two distinct periods. The values of these two periods are uniquely determined by the values of s and AN/B or Ω; conversely s and Ω, and hence μ and f_M are uniquely determined by the values of the periods. The main object of the jump card experiment, described in this paper, is to determine the two periods of the initial angular oscillations of a shell, fired horizontally from a gun. As Ω depends only on the spin N (known in terms of the muzzle velocity) and the moments of inertia, there is in general an independent check on the observation.* By firing the shell at a series of different muzzle velocities, values of f_M are determined for different values of the variable v/a, resulting in the curves of § 1.21.

1.32. The success of the experiments depends entirely on the occurrence of accidental disturbances at the muzzle, in order to produce oscillations of sufficient amplitude to be measurable. The methods of observation used were capable of giving accurate results, provided that the maximum yaw exceeded 1 degree. In the actual trial, no round was fired which developed a maximum yaw of less than 2 degrees, and it is probable that with almost any type of shell the initial disturbance would be sufficient for observations of this nature to be made. It may be noticed that, for a given initial disturbance, the amplitude of the oscillations is greater, the smaller the value of s, until, as s approaches and becomes smaller than the value unity, the amplitude of the oscillations increases very rapidly. For this reason it was at first considered preferable to deal with a shell and gun for which s was only just greater than unity, but the experiments described in this paper indicate that a value of s in the neighbourhood of $1 \cdot 5$ will give the best general results.

It is to be expected *a priori*, and is confirmed by the experiment, that the initial yaw of a shell, on leaving the muzzle of a gun, is very small, and that the angular oscillations are due mainly to an initial angular velocity about a transverse axis. The shell is completely unstable under the very large pressures of the powder gases on its base, so that as soon as it is released from the barrel it is disturbed from its position of unstable equilibrium by an amount, and in a direction, which depend largely on accidental circumstances.† The pressure of the powder gases probably continue to influence the motion over a short interval after the shell has left the gun, but the whole effect on the shell must approximate to that of an impulsive couple about a transverse axis.

The angular motion of the shell, for some distance from the muzzle, approximates, therefore, to the type of motion of a spinning top known as rosette motion, in which the axis of the top passes periodically through the vertical.

* This check is especially important in the case of shells of type II., as the shift of the lead block on firing alters the values of the dynamical constants as determined by laboratory experiments (§ 2.2).

† [*Note added July* 31, 1920. In view of further analysis of the initial circumstances of shells in this trial, this account of the matter is probably incomplete.]

1.33. *Differences Between the Shell and Top Movements.*—We now proceed to consider the factors, so far neglected, which cause the angular motion of the shell to differ from that of the corresponding top. These may be enumerated as follows :—

(1) The effect of the cross-wind force in causing the centre of gravity to follow a curve of helical type.
(2) The effect of the force components denoted in § 1.11 and § 1.12 by H, J, and K.
(3) The effect of the diminution of forward velocity caused by the drag.
(4) The effect of gravity.

These effects will be considered in turn.

1.331. The angular oscillations of the shell give rise to a cross-wind force, which varies in magnitude and direction as the yaw varies, and this modifies the straight line motion along the direction of projection into motion of a helical type. If this helical motion could be observed with accuracy it would give valuable data for the cross-wind force coefficient f_L, but unfortunately the amplitude of the oscillations is too small to allow of this. Hence the most important effect, from the point of view of these experiments, is the reaction of the sideways motion of the centre of gravity on the angular oscillations of the shell. This helps to damp out the oscillations.

1.332. The yawing moment factor H has a similar damping effect as it is always opposed to the transverse angular velocity. While the effect of the former factor is to damp the slow period oscillation and slightly augment the quick oscillation, this latter has exactly the reverse effect. In combination, they, in general, damp out the oscillations of both periods. For the 3-inch shells, used in this trial, the yawing moment damping factor is of greater importance than the cross-wind force damping factor, and the general effect is to diminish the maximum values of the yaw, and at the same time to convert the initial rosette motion into the slower steady precessional motion.* The force component, J, due to the spin, has no appreciable effect on the angular motion, but the corresponding couple K might act as a small additional damping factor.

1.333. The head resistance or drag slowly diminishes the forward velocity, and so increases the stability factor s, by diminishing μ. The change in s diminishes the amplitude of the oscillations to a limited extent, and so assists the other damping factors.

1.334. Gravity affects the angular motion of the axis of the shell by producing curvature in the trajectory. In taking account of the gravity effect it is necessary to

* There are two possible types of steady precessional motion at constant yaw, one with a quick and the other with a slow precessional velocity.

refer the motion to axes moving with the tangent to the trajectory (see § 3.2). The effect is quite insignificant over the range covered by the present trial, but becomes of importance at later stages of the trajectory, where it is responsible for producing the drift.*

It is convenient to illustrate this effect by considering a simple case of steady motion.

1.34. *An Illustration of the Gravity Effect.*—Let the centre of gravity of a shell be constrained to move through air at a constant speed v, in a vertical circle (fig. 7), the inclination of the path to the horizontal being θ at any instant. Thus v and $d\theta/dt$ are constant. There is a possible steady motion in which the axis OA always lies in the plane through OP perpendicular to the plane of the circle, the angle AOP (δ) being constant. The couple M tending to increase δ will also be constant, so that the contemplated motion is the same as the steady motion of a top making an angle $\frac{1}{2}\pi+\delta$ with the vertical which corresponds to the normal to the plane of the circle. The angular velocity of the axis about this normal is $-\theta'$; the value of δ as given by the ordinary formula for the steady motion of a top under these conditions† is

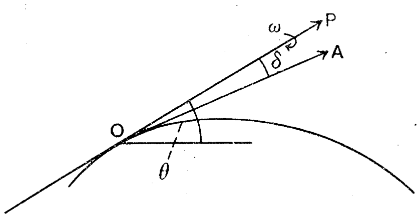


Fig. 7.

$$-\mathrm{AN}\theta' \cos\delta + \mathrm{B}\theta'^2 \sin\delta\cos\delta = \mathrm{M} = \mu\sin\delta. \tag{1.341}$$

If θ' is not too large and μ is not too small, a possible value of δ is small; we may now regard μ as independent of δ, and the equation then reduces to

$$\delta = -\mathrm{AN}\theta'/\mu = -4s\theta'/\Omega, \tag{1.342}$$

the term neglected being of order δ^3. When a shell is moving freely the angular velocity θ' increases, and the linear velocity diminishes up to a point beyond the vertex of the trajectory. If the initial motion is identical with the above steady motion, this will cause the couple M to diminish, so that the axis of the shell will lag behind its position in the steady motion. This lag gives rise to a component angular velocity of the axis tending to increase the yaw δ, until a state of relative equilibrium is reached, in which the yaw is slightly less than its equilibrium value, and the axis lags slightly behind (*i.e.*, above) the tangent OP. When the velocity is high and the spin N not too large, M is large and the true position of the axis lies very near the equilibrium position. It will be shown in fact, in Part IV., that the assumption

* For a shell whose spin and direction of motion are related like a right-handed screw the drift is to the right of the plane of fire.

† See, *e.g.*, ROUTH, 'Rigid Dynamics,' vol. II., Art. 207.

that OA is level with OP, and that δ is given by (1.342), lead to a determination of the drift which is sufficiently accurate for all trajectories of elevation less than 30 degrees. The drift is produced by the cross-wind force resulting from the above value of the yaw.

In the neighbourhood of the vertex of a trajectory of high elevation, both the velocity and the couple M become very small, so that δ becomes large. A calculation has been made, by a step-by-step process, of the angular motion and drift of a shell fired at an elevation of 70 degrees. The yaw, soon after the vertex, reaches the value 60 degrees, while the axis lags behind the tangent to the path by more than 45 degrees.

1.35. The effect of gravity as described in the last section completes the list of factors which have an appreciable effect on the motion, and it remains to consider the way in which they combine. It will be shown in § 3.2 that the motion of the axis of a stable shell is determined, to a good approximation, when the yaw is not too large, by a linear differential equation of the second order. The effect of gravity is to produce the type of motion described in § 1.34, given the proper initial conditions in which the yaw and its rate of increase are both very small. The complete motion under arbitrary initial conditions may be obtained by superposing the appropriate type of initial oscillatory motion, which is unaffected by gravity. The superposed oscillations will ultimately be damped out, leaving the motion of the last section only. The motion of the centre of gravity will be appreciably affected by alteration of the initial conditions only in so far as they produce a certain small sideways displacement and velocity (§ 4.2), and increase the drag to an extent which is not yet known.

More detailed results are reserved for Part IV., following the discussion of the mathematical theory. Actual examples of the observed motion of the shell's axis can be studied in fig. 14.

Part II.—Details of the Experimental Arrangements and Material.

§ 2.0. *General Arrangements.*

We propose, in this part, to explain the details of the experiments in so far as is necessary to enable the reader to understand the method used, and to form an estimate of the accuracy obtained, or capable of being obtained, in this manner.

The experiments were carried out as the weather served in January and February, 1919, four different types of 3-inch shells being fired, at various velocities, from each of two differently rifled guns. The constants of the shells used are given in Table III.,

TABLE III.—Mean Values of the Dynamical Constants of the Shells used, Determined before Firing.

Types I., II. and III., Form A. } (See fig. 6.)
Type IV., Form B.

Type of shell.	Length, inches.	Weight, lb.	Distance of centre of gravity from base, inches.	Axial moment of inertia, A, lb. (in.)^2.	Transverse moment of inertia, B, lb. (in.)^2.	B/A.
I. (Normal) . . .	11·53	14·09	4·727	18·37	143·9	7·83(5)
II. (Centre of gravity forward)	11·53	16·31	5·124	19·20	165·0	8·59
III. (Centre of gravity back)	11·53	16·48	4·203	18·93	129·5	6·84
IV. (Shells with pointed nose)	13·15	14·62	4·965	18·71	166·2	8·89

and details of the groups fired are given in Table IV. The distance available between the firing point and the sea at Portsmouth is rather less than 600 feet. The motion of the shell was recorded over this range, within which the effects of gravity are fairly small and the path of the shell not widely different from a straight line. To achieve this the shell was fired through a series of millboard pistol targets, 2 feet square, about $\frac{1}{20}$ inch thick, which were fastened approximately at right angles to the path of the shell, at suitable distances from the muzzle.* The plane of the card was carefully adjusted, and it is probable that in no case did the angle between the path of the shell and the plane of the card differ from a right angle by as much as two degrees. As errors up to four degrees do not affect the shape and position of the hole in the card, which determine the position of the axis and the centre of gravity of the shell at the moment of impact, it may be assumed that in every case

* For the gun whose rifling made one complete turn in a length of 40 diameters of the bore (rifled 1 in 40) ten cards were used, placed approximately at 60-foot intervals, the first card being 50 feet from the muzzle. For the gun rifled 1 in 30 twelve cards were used, the first seven being at 30-foot intervals and the later cards at 60-foot intervals as before. The distance of the cards from the muzzle of the gun was determined with a probable error of 1 inch.

TABLE IV.—Showing Groups of Rounds Fired.

The types of shell are numbered I.–IV., and the shells of each type are numbered 1, 2, 3, ... in the order of firing.*

GUN rifled one turn in 40 diameters of the bore.

Group.	Mean muzzle velocity for group, f.s.	Remarks.	Group.	Mean muzzle velocity for group, f.s.	Remarks.
Type I.; shells of form A; centre of gravity normal.					
I. 11–14	922	Stable	I. 15, 16	2130	Stable †
I. 8–10	1072	Unstable	I. 1–4	2167	Stable
I. 17, 18	1312	Unstable	I. 19	2272	Stable
I. 5–7	1565	Just stable	I. 20, 21	2346	Stable
Type II.; shells of form A; centre of gravity forward.					
II. 8–10	934	Stable	II. 5–7	1585	Stable
II. 11–13	1107	Unstable	II. 1–4	2024	Stable
II. 14–16	1334	Unstable			
Type III.; shells of form A; centre of gravity back.					
III. 8–10	931	Stable	III. 5–7	1583	Just stable
III. 11–13	1077	Unstable	III. 1–4	2025	Stable
III. 14–16	1312	Unstable			
Type IV.; shells of form B; centre of gravity normal.					
IV. 10–12	884	Very unstable	IV. 1–6	2130	Very unstable
IV. 7–9	1553	Very unstable			

* Only the stable groups are analysed in this report. For a specimen yaw curve in an unstable case, see fig. 12.

† Fired with cards on the far screens only, to determine by comparison the effect of the impacts on the cards.

TABLE IV. (continued).

GUN rifled one turn in 30 diameters of the bore.

All groups were stable.

Group.	Muzzle velocity, f.s.	Group.	Muzzle velocity, f.s.	Group.	Muzzle velocity, f.s.
Type I.					
I. 22–24	1119	I. 25, 26	1326	I. 27, 28	1563
Type II.					
II. 17–19	1119	II. 24	1292	II. 22, 23	1589
Type III.					
III. 17–19	1119	III. 20, 21	1292	III. 22, 23	1567
Type IV.					
IV. 21–23 IV. 13–15	900 1078	IV. 16–18 IV. 19, 20*	1547 1547	IV. 24–26	2121

the centre of gravity of the shell was moving normally to the card.† Thus the angle actually recorded by the shape of the hole in the card is the true yaw of the

* Fired with cards on the far screens only, to determine by comparison the effect of the impacts on the cards.

† The angular motion of the axis of the shell is comparatively so slow that it can be ignored during the interval in which the shell is passing through a card. For instance, with the shells used in this trial the change in ϕ, the orientation of the yaw, is never as much as $3\frac{1}{2}$ degrees during the complete passage through the card, and the change in δ never as much as 8 minutes. These quantities are of the same order as the errors of observation and may be ignored. Thus the shell can correctly be regarded as equivalent for cutting purposes to its circumscribing cylinder (of indefinite length) whose generators are parallel to the direction of motion of the centre of gravity.

If the direction of motion is normal to the plane of the card at the moment of impact, a certain hole will be cut in the card, whose shape will be precisely that of the normal cross-section of this circumscribing cylinder. But if the card is tilted through a small angle τ about any axis in its own plane, the hole made by the shell will be the same as the cross-section of the supposed cylinder by the plane of the card shell, that is, the angle between the axis of the shell and the direction of motion of its centre of gravity.

On each card there was marked, by methods which need not be particularised, (*a*) the vertical, (*b*) a reference point from which the point of aim for each round could be deduced. The probable error in the marking of the vertical was negligible compared to the other errors of observation. The probable error* in each co-ordinate of the point of aim was about $0 \cdot 2$ inches.

Times of flight from the muzzle to each card were not directly observed, but the mean velocity of the shell over a suitable interval of the range was observed for each round with two standard Boulangé chronographs. These were sometimes used as a pair—in these cases their readings were in good agreement—and sometimes separately, at opposite ends of the range, to determine the loss of velocity, and so an approximate value for the average coefficient of the drag. From the data so obtained, the muzzle velocity and the times of flight from the muzzle to each screen were calculated by the usual ballistic methods to a nominal accuracy of 1 f.s. and 10^{-4} second, respectively. It is improbable that any of these quantities are appreciably in error to the order of accuracy required by the rest of the experiment. A check on the calculated muzzle velocity is provided by the observations, for a discussion of which the reader should refer to § 4.1.

§ 2.1. *Measurement of the Holes in the Cards.*

It is now necessary to deduce, from the position and shape of a hole in any card, the position of the axis and centre of gravity of the shell at the moment of passing the card. This can usually be done with considerable accuracy. It has been found that at all velocities less than 1600 f.s., and often at higher velocities, the hole has the form shown diagrammatically in fig. 8, and by photographs of actual examples in fig. 8A.

Inside the outer circumference ABA′B′ of the hole, a considerable amount of bruised and partly torn card QQQ is left, which is still attached to the untouched part. It is found that, when the edges of this part are flattened out, they always define with some accuracy a circle of diameter $2 \cdot 40$ inches. A stiff paper circle of this diameter can be fitted to the hole with such certainty that its centre is seldom in doubt by more than $0 \cdot 01$ or at most $0 \cdot 02$ inches.

in its tilted position. The dimensions of such a hole will only differ from those of normal impact by terms of order $d(1 - \cos \tau)$, where d is any dimension of the hole. Such second-order terms are completely negligible if $\tau < 4$ degrees. Thus in all cases the shell may be regarded as cutting the hole in the card as if the direction of motion of its centre of gravity is normal to the plane of the card at the moment of impact.

* Throughout this paper "probable error" is used with its technical meaning, see *e.g.*, BRUNT, 'The Combination of Observations,' p. 30.

The external form of the shells used in the trial is shown in fig. 6. It will be observed that at the junction of the body of the shell and the fuze or plug there is a distinct cutting edge of plan diameter 2·402 inches. It is clear, therefore, that when the impact takes place, a circle of cardboard, 2·40 inches in diameter, is punched out and cleanly removed by this edge; the greater part of the circumference of this inner circle is usually removed by the subsequent passage of the body of the shell, which cuts the complete hole, but enough remains, in a bruised state, for yaws that are not

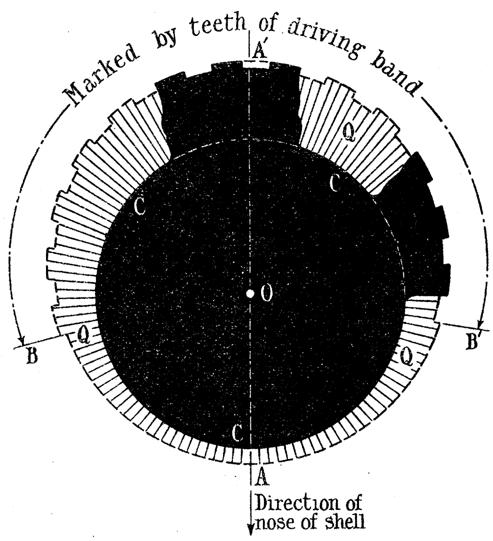


Fig. 8. Diagrammatic sketch of a typical hole, for a yaw between 1 degree and 4 degrees, when the velocity is low or medium.

CCC. Inner circle—radius 2·40 inches, centre O.
ABA′B′. Outer circumference of hole.
QQQ. Bruised part of card.
AA′. Axis of symmetry or greatest diameter of hole.
BA′B′. Circumference cut by teeth of driving band.
BAB′. Ditto cut by nose or shoulder of shell.

The lengths AA′ (3·16 inches in figure) and OA′ (1·80 inches) each serve to determine the size of the yaw.

The values of the yaw corresponding to the above values are 1·6 degrees and 1·8 degrees respectively, mean 1·7 degrees.

too large, to define the position of the centre of this section of the shell at the moment of impact on the card.

It follows, therefore, that there are two distinct methods by which the value of the yaw δ can be determined. In the first place, there is a unique relation between the greatest diameter of the hole (AA′, fig. 8) and the value of δ; secondly, there is a unique relation between OA′ and δ. These relations can be tabulated numerically when the plan dimensions of the shell are known, and the value of δ corresponding to any measured length AA′ or OA′ read off.

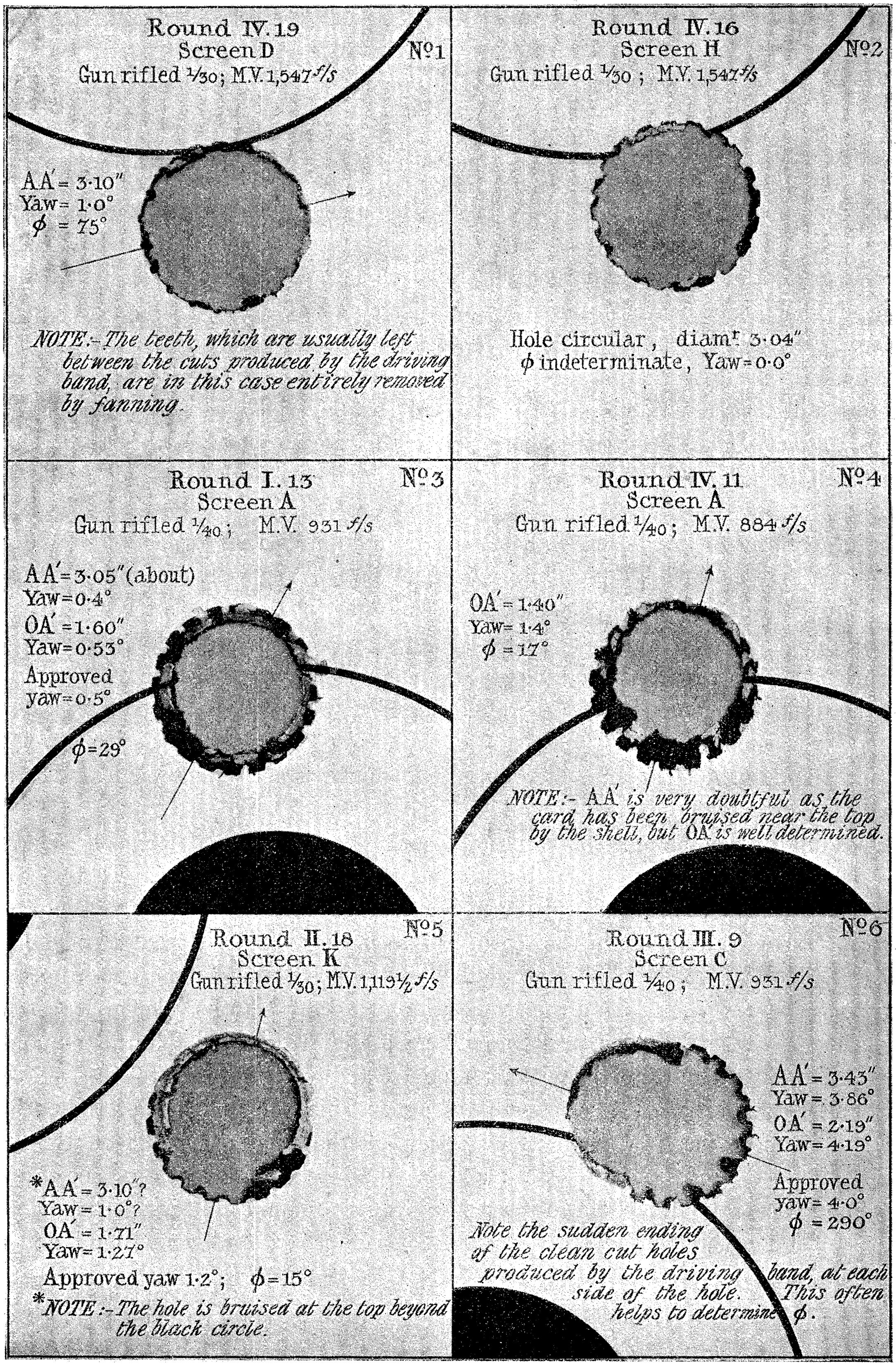

Round IV. 19
Screen D
Gun rifled 1/30; M.V. 1,547 f/s
No. 1
AA′ = 3·10″
Yaw = 1·0°
φ = 75°
NOTE:—The teeth, which are usually left between the cuts produced by the driving band, are in this case entirely removed by fanning.
Round IV. 16
Screen H
Gun rifled 1/30; M.V. 1,547 f/s
No. 2
Hole circular, diamr. 3·04″
φ indeterminate, Yaw = 0·0°
Round I. 13
Screen A
Gun rifled 1/40; M.V. 931 f/s
No. 3
AA′ = 3·05″ (about)
Yaw = 0·4°
OA′ = 1·60″
Yaw = 0·53°
Approved yaw = 0·5°
φ = 29°
Round IV. 11
Screen A
Gun rifled 1/40; M.V. 884 f/s
No. 4
OA′ = 1·40″
Yaw = 1·4°
φ = 17°
NOTE:—AA′ is very doubtful as the card has been bruised near the top by the shell, but OA′ is well determined.
Round II. 18
Screen K
Gun rifled 1/30; M.V. 1,119½ f/s
No. 5
*AA′ = 3·10″?
Yaw = 1·0°?
OA′ = 1·71″
Yaw = 1·27°
Approved yaw 1·2°; φ = 15°
*NOTE:—The hole is bruised at the top beyond the black circle.
Round III. 9
Screen C
Gun rifled 1/40; M.V. 931 f/s
No. 6
AA′ = 3·43″
Yaw = 3·86°
OA′ = 2·19″
Yaw = 4·19°
Approved yaw = 4·0°
φ = 290°
Note the sudden ending of the clean cut holes produced by the driving band, at each side of the hole. This often helps to determine φ.

Fig. 8A.

To determine the value of ϕ,* it is necessary to measure the angle between AA′ and the vertical recorded on the card. The direction of AA′ must be determined by eye from the considerations that it is (1) the greatest diameter, (2) the axis of symmetry of the hole, and (3) that it must pass through O, which is located as the centre of a paper circle fitted into the inner hole.

By proceeding in this manner it was found that the values of δ could be nearly always determined with confidence by at least one method and often by both. When both methods were available the agreement in the resulting values of δ was in general good; the average difference between them in *all* cases for shells of type I. (99 in number), in which both measurements were available and both appeared to be *a priori* reliable, was 0·20 degrees. These cases were simply taken as a sample. The general features of the agreement were the same for all types. We may therefore fairly assert that the probable error of any determination of δ is something less than 0·2 degrees. The use of the measurement OA′ is of special importance for small values of δ, and in fact alone makes their accurate determination possible.

The probable error in the determination of ϕ is not quite so easy to estimate, as there is no alternative method of determining ϕ. The method is clearly theoretically sound, and the errors can only arise from faulty estimations of the symmetry of the hole. By making a number of independent determinations for the same hole, with proper precautions against a biassed judgment, and comparing their consistency, it appeared that the probable error of any determination of ϕ was less than $2\frac{1}{2}$ degrees, unless the yaw was small (less than 0·8 degrees, say). As the yaw approaches zero, the errors in the determination of ϕ increase rapidly until, when the yaw is less than 0·2 degrees, ϕ cannot be determined at all.

Proceeding in this manner the values of δ and ϕ were tabulated for each round for the values of the time corresponding to the position of each card. If the above estimates are correct it is doubtful if the accuracy obtained could be much improved on without a radical change in the method of recording the position of the shell.

2.11. When the yaw has been determined, and the position of the centre of gravity on the axis of the shell is known, its position along AA′ can be calculated from the dimensions of the shell. The position of AA′ on the card is well determined, and so the position of the centre of gravity can be located with respect to the reference point, and so with respect to the point of aim. This part of the determination is considerably more accurate than the location of the reference point on the card. The path of the centre of gravity for a small number of rounds was measured up in this manner; the results of the discussion (§ 4.2) are mainly null, in agreement with theory. The measurements were therefore not completed for every round and are not given here.

* The angle ϕ denotes the angle between the plane of yaw OAP and the vertical plane through OP. See fig. 10, p. 332.

§ 2.2. *Determination of the Dynamical Constants.*

All the shells used in these experiments were weighed before firing, and their overall lengths were measured. The variations from shell to shell were small, and the mean values given in the tables may be assumed to be correct for all purposes. No appreciable change in these quantities is likely to occur on firing.

The moments of inertia were determined, before firing, for a selection of about 25 per cent. of the shells of each type. The probable error of any determination was about 1 part in 2000. The mean values for the different types of shell are given in the table. The extreme variation of any transverse moment of inertia from the mean was $1 \cdot 8$ per cent., and of the axial moment of inertia was $0 \cdot 8$ per cent. The errors in assuming that the mean value of the sample is the correct value for each round may therefore be appreciable at times, but should not seriously affect the final mean results. The general accuracy of the experiments was, contrary to expectation, sufficient to warrant the refinement of determining and using the individual values for each shell.

The centres of gravity were also determined, before firing, for the same selection of shells, and the mean value of the distance of the centre of gravity from the base is given, in the same table, for each type. The determination was made with a probable error of $0 \cdot 003$ inches. The values were fairly constant for the shells of any one type, the extreme variation from the mean being $0 \cdot 022$ inches.

It is by no means certain *a priori* that the values of A and B and the position of the centre of gravity may not be changed appreciably in some of the shells by the stresses set up when the gun is fired. No change is at all likely in the empty shells of types I. and IV., or in the bodies of the other shells; they may be confidently relied upon not to be stressed beyond their elastic limit; but the lead and wood filling in the shells of types II. and III. is decidedly suspect. To test this point, two shells of each of the types II. and III., after the determination of their dynamical constants, were fired* over water for recovery, and their constants were then re-determined. In the case of the shells of type III., with a filling of lead at the back and wood in front, there was no appreciable change. In the case of the shells of type II. with lead in front and wood behind, the wood block, as might have been expected, was crushed, and the lead had moved back about three inches in the case of the high velocity and one inch in the case of the low. The axial moments of inertia, A, were unaltered, but the transverse moments of inertia B and the positions of the centre of gravity were of course seriously affected. It was found, however, that the observed changes in both could be satisfactorily accounted for by the observed movement of the lead block, of weight $1 \cdot 9$ lb. When the centre of gravity of the shell of type II. is $4 \cdot 727$ inches from the base, so that it coincides with the centre of gravity of a shell of

* One of each type at a muzzle velocity of 1950 f.s. and one at 1530 f.s.

type I., the value of B is 145·7 lb. (inch)^2. Neglecting the effect of the wood, suppose that the lead plug is x inches further forward. In such a case

$$B = 145 \cdot 7 + 1 \cdot 9x^2. \tag{2.21}$$

If, moreover, l is the distance of the centre of gravity from the base in inches, then

$$l = 4 \cdot 727 + 0 \cdot 117x. \tag{2.22}$$

The altered position of the centre of gravity can therefore be recovered by calculation, if the altered value of B can be deduced from the observations. This is, in fact, the case (see § 4.1), so that even for shells of type II. the dynamical constants of the shells after firing are satisfactorily certain.

§ 2.3. *Possible Disturbing Factors.*

There are two further possible causes of error which we have not yet mentioned. These are (1) the wind, and (2) the impulsive action between the shell and the card.

2.31. *The Effect of Wind.*—Since we are studying the motion of the shell under the force system impressed by the air, we are concerned solely with the motion of the projectile *relative to the air*, but we can only observe, by means of jump cards, the motion of the projectile *relative to the ground.*

If the strength and the direction of the wind are known, it is an easy matter to convert the observed values of the size and orientation of the yaw, and the observed motion of the centre of gravity, into the corresponding quantities for the motion relative to the air. It is, however, very difficult to determine what is the strength of the wind, at the moment of firing, only a few feet above the ground. It is, therefore, necessary to carry out jump card trials in calm weather. During the experiments the wind exceeded 10 f.s., only at the moments of firing three rounds, and was usually only 5 or 6 f.s. at 20 feet above the ground. Its strength near the ground will have been still less, and its effects may therefore be neglected.

2.32. *The Impulsive Action between the Shell and the Card.*—When the experiments were started it was not expected that the effect of the cards would be decidedly bigger than the probable random errors of the results. This, however, appears to be the case. A limited amount of evidence, for determining the necessary correction, is supplied by the few comparative rounds fired without cards on the nearer screens. Such comparative rounds would have been included in all, or at least the majority, of the groups, if their importance had been realised earlier. The evidence supplied by the comparative rounds was carefully analysed, and was supplemented, after the conclusion of the trial, by determination of the *magnitude* of the impulse between the cards and the shells by observation of the extra loss of velocity so caused. The magnitude of a single impulse, at not too great a value of the yaw, probably has the values

$$14 \cdot 3 \text{ foot-poundals at } 2470 \text{ f.s.},$$

$$8 \cdot 9 \text{ foot-poundals at } 1140 \text{ f.s.};$$

the values at other velocities may be roughly obtained by linear interpolation.

The effect on the observed motion of the axis due to an impulsive couple was calculated, and it was found that rough values could be assigned for the magnitude of the impulsive couple acting at any card. On calculating the total effect on the observed value of s it was found that the probable correction required varied from $2\frac{1}{2}$ to $4\frac{1}{2}$ per cent. in the various groups. This correction was applied before constructing Table I. and figs. 4 and 5. The figures of Table II. have not been corrected for this effect as their accuracy is not great enough to make it worth while to do so.

PART III.—METHODS OF OBTAINING AND SOLVING THE EQUATIONS OF MOTION OF A SPINNING SHELL.

§ 3.0. *Introductory.*

On the assumptions discussed in Part I. the equations of motion of a spinning shell can be written down at once by the rules of rigid dynamics. Three different types of these equations will be found of use in practice, all of which may be obtained most simply as special cases of the vector equations of motion of the shell, referred to axes rotating in the most general manner. The use of the vector notation, in the initial stages of the discussion, has the further advantage of showing most clearly the meaning of the various terms, and of presenting the results in a symmetrical form.

In order to simplify the general equations, the only components of the force system impressed by the air, retained in the initial discussion, are R, L, M, and the spin-retarding couple I ($= \mathrm{AN}\Gamma$). The remaining components are of less importance and will be inserted later on in § 3.5.

After obtaining the general equations the three special types are deduced. They may be described as follows:—

Type α.—Equations in terms of direction cosines, referred to axes moving with the tangent to the corresponding plane trajectory.

Type β.—Equations in terms of direction cosines or spherical polar co-ordinates, referred to axes moving with the tangent to the actual twisted trajectory.

Type γ.—Equations similar to the equations of energy and angular momentum of a top (spherical polar co-ordinates), referred to the axes used for type β.

In each case the equations obtained are simplified by certain approximations, and the results are suitable for use only under certain conditions. Equations of type α are valid when the shell is sufficiently stable and the yaw is small; type β when the shell has settled down to a non-periodic motion in which the yaw may be large, the initial oscillations being damped out; and type γ when the motion of the centre of gravity is nearly rectilinear.

Equations of these types cannot be solved exactly, and the method of approximation used to obtain a solution is different in each case. The equations of type α are used for the analysis of the jump card experiments, for all sufficiently stable rounds, and could be used to compute the entire motion in any trajectory whose initial elevation is less than 45 degrees. Equations of type β have been used to compute the latter part of a twisted trajectory at an elevation of 70 degrees. Equations of type γ have a limited application in analysing the jump card records for rounds which are nearly or quite unstable.

3.01. *Note on the Vector Notation.*—All letters which represent vector quantities will be in clarendon type, to distinguish them from scalar quantities in the ordinary type. The three components of any vector $\mathbf{A}$, referred to right-handed rectangular axes 1, 2, 3, are written A_1, A_2, A_3.

If $\mathbf{A}$ and $\mathbf{B}$ are two vectors, their vector product is denoted by $[\mathbf{A}\,.\,\mathbf{B}]$. This represents the vector whose components are

$$(A_2B_3-A_3B_2), \qquad (A_3B_1-A_1B_3), \qquad (A_1B_2-A_2B_1).$$

It is perpendicular to the plane containing the two vectors in the direction of the axis of the right-handed screw, which turns from $\mathbf{A}$ to $\mathbf{B}$, its modulus being equal to the product of the moduli of $\mathbf{A}$ and $\mathbf{B}$ into the sine of the angle between them. The scalar product of the two vectors is written $(\mathbf{A}\,.\,\mathbf{B})$, and is equal to the scalar quantity

$$A_1B_1+A_2B_2+A_3B_3;$$

it is also equal to the product of the moduli of $\mathbf{A}$ and $\mathbf{B}$ into the cosine of the angle between them, being positive when this angle is acute. For simplicity, we denote $(\mathbf{A}\,.\,\mathbf{A})$ by $(\mathbf{A})^2$, which is equal to the square of the modulus of $\mathbf{A}$.

Constant use is made of the following identities:—

$$(3.011) \qquad [\mathbf{A}\,.\,\mathbf{A}] = 0,\ ([\mathbf{A}\,.\,\mathbf{B}]\,.\,\mathbf{A}) = 0.$$

$$(3.012) \qquad [[\mathbf{A}\,.\,\mathbf{B}]\,.\,\mathbf{C}] = (\mathbf{A}\,.\,\mathbf{C})\,\mathbf{B}-(\mathbf{B}\,.\,\mathbf{C})\,\mathbf{A}.$$

$$(3.013) \qquad ([\mathbf{A}\,.\,\mathbf{B}]\,.\,[\mathbf{B}\,.\,\mathbf{C}]) = (\mathbf{A}\,.\,\mathbf{B})\,(\mathbf{B}\,.\,\mathbf{C})-(\mathbf{B})^2\,(\mathbf{A}\,.\,\mathbf{C}).$$

§ 3.1. *The General Vector Equations of Motion.*

We take a system (1, 2, 3) of right-handed axes of reference, see fig. 9, whose origin is O, the centre of gravity of the shell, and whose angular velocity at any instant is represented by the vector $\mathbf{\Theta}$, with components Θ_1, Θ_2, Θ_3. The direction of the axis of the shell OA is represented by the unit vector* $\mathbf{\Lambda}$, and the direction of motion of the centre of gravity by the unit* vector $\mathbf{X}$.

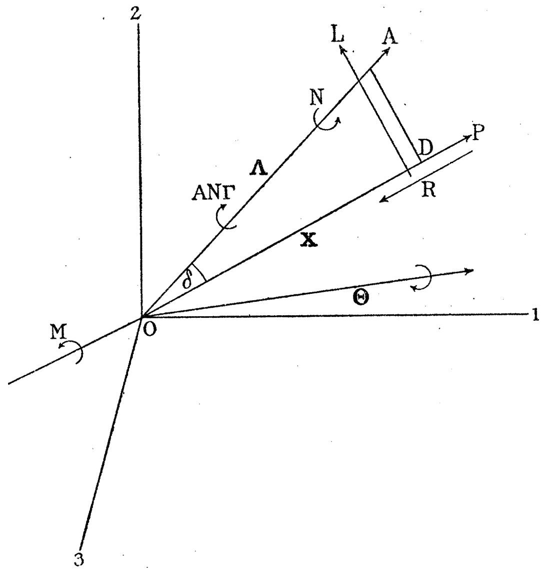


Fig. 9.

With the notation already introduced in Part I., the total angular momentum of the shell can be expressed as the sum of two vectors :—

(i.) The angular momentum about OA, AN$\mathbf{\Lambda}$;

(ii.) The total angular momentum about a transverse axis.

If the total angular velocity about a transverse axis is w, the angular momentum is Bw, and is equal to the moment of momentum of a particle whose mass is B and whose distance from O is represented by the vector $\mathbf{\Lambda}$. Now the actual velocity of such a particle relative to O is $\mathbf{\Lambda}' - [\mathbf{\Lambda} . \mathbf{\Theta}]$, and therefore its moment of momentum about O is

$$B\{[\mathbf{\Lambda} . \mathbf{\Lambda}'] - [\mathbf{\Lambda} . [\mathbf{\Lambda} . \mathbf{\Theta}]]\}.$$

The total angular momentum, $\mathbf{H}$, of the shell about O is therefore given by the equation

(3.101) $$\mathbf{H} = AN\mathbf{\Lambda} + B\{[\mathbf{\Lambda} . \mathbf{\Lambda}'] - [\mathbf{\Lambda} . [\mathbf{\Lambda} . \mathbf{\Theta}]]\} ;$$

using (3.012) this becomes

(3.102) $$\mathbf{H} = AN\mathbf{\Lambda} + B\{[\mathbf{\Lambda} . \mathbf{\Lambda}'] - (\mathbf{\Lambda} . \mathbf{\Theta})\,\mathbf{\Lambda} + \mathbf{\Theta}\}.$$

* *I.e.* $(\mathbf{\Lambda})^2 = (\mathbf{X})^2 = 1$.

The force components that we propose to include at this stage are R, L, M, and ANΓ. To simplify the algebra we write*

$$L = \kappa m v \sin \delta, \qquad M = \mu \sin \delta.$$

The various components can then be represented by the following vectors:—

(i.) The drag R, by the vector $-R\mathbf{\Lambda}$;

(ii.) The cross-wind force L, by the vector† $\kappa m v \{\mathbf{\Lambda} - \mathbf{X} \cos \delta\}$;

(iii.) The couple M, by the vector† $\mu [\mathbf{X} . \mathbf{\Lambda}]$;

(iv.) The couple ANΓ, by the vector $-AN\Gamma\mathbf{\Lambda}$.

The complete equation for the angular motion is therefore

$$\mathbf{H}' - [\mathbf{H} . \mathbf{\Theta}] = \mu [\mathbf{X} . \mathbf{\Lambda}] - AN\Gamma\mathbf{\Lambda}, \tag{3.103}$$

where **H** is given by (3.102). Taking the scalar product of both sides of (3.103) into $\mathbf{\Lambda}$, we obtain, with the help of (3.011)–(3.013),

$$N' = -N\Gamma. \tag{3.104}$$

After substituting for N′, equation (3.103), written in full, reduces to

$$\begin{aligned} AN\mathbf{\Lambda}' + B[\mathbf{\Lambda} . \mathbf{\Lambda}''] - 2B(\mathbf{\Lambda} . \mathbf{\Theta})\mathbf{\Lambda}' - B(\mathbf{\Lambda} . \mathbf{\Theta}')\mathbf{\Lambda} + B\mathbf{\Theta}' \\ - AN[\mathbf{\Lambda} . \mathbf{\Theta}] + B(\mathbf{\Lambda} . \mathbf{\Theta})[\mathbf{\Lambda} . \mathbf{\Theta}] = \mu[\mathbf{X} . \mathbf{\Lambda}]. \end{aligned} \tag{3.105}$$

3.11. *The Equations of Motion of the Centre of Gravity.*—The velocity of the centre of gravity is represented by the vector $v\mathbf{X}$, and its acceleration is therefore represented by the vector

$$\frac{d}{dt}\{v\mathbf{X}\} - v[\mathbf{X} . \mathbf{\Theta}].$$

In addition to the drag and cross-wind force impressed by the air, we shall suppose that gravity is acting on the shell.

* The mass and velocity of the shell are m and v respectively. For the rest of the notation see § 1.31.

† If a perpendicular AD be drawn from A to OP, DA is parallel to the direction of the cross-wind force L, and its length is sin δ, if OA is of unity length. The vector DA is equal to the difference of the vectors OA and OD, so that it is equal to $\mathbf{\Lambda} - \mathbf{X} \cos \delta$. Hence $\{\mathbf{\Lambda} - \mathbf{X} \cos \delta\}/\sin \delta$ is the unit vector parallel to the cross-wind force. Similarly $[\mathbf{X} . \mathbf{\Lambda}]/\sin \delta$ is the unit vector normal to the plane AOP *i.e.*, parallel to the axis of the couple M. It is easy to verify, with the help of (3.012), that

$$\mathbf{\Lambda} - \mathbf{X} \cos \delta = [[\mathbf{X} . \mathbf{\Lambda}] . \mathbf{X}].$$

The acceleration due to gravity is represented by the vector $\mathbf{G}$, whose modulus is g.* Under these conditions the vector equation of motion of the centre of gravity is

$$(3.111)\qquad \frac{d}{dt}\{v\mathbf{X}\}-v[\mathbf{X}.\mathbf{\Theta}] = -\frac{R}{m}\mathbf{X}+\kappa v\{\mathbf{\Lambda}-\mathbf{X}\cos\delta\}+\mathbf{G}.$$

Taking the scalar product of both sides into $\mathbf{X}$, equation (3.111) reduces to

$$(3.112)\qquad v' = -R/m+(\mathbf{G}.\mathbf{X}).$$

On substituting this value of v' in (3.111), and dividing by v, we obtain

$$(3.113)\qquad \mathbf{X}'-[\mathbf{X}.\mathbf{\Theta}] = \kappa\{\mathbf{\Lambda}-\mathbf{X}\cos\delta\}+\{\mathbf{G}-(\mathbf{G}.\mathbf{X})\mathbf{X}\}/v.$$

Equations (3.104), (3.105), (3.112), and (3.113) determine the motion completely.

§ 3.2. *Equations of Motion of Type* α.

When a shell is initially sufficiently stable, and leaves the muzzle so that its initial disturbance is small, it will be shown† that the axis OA and the direction of motion OP deviate, at any time t, by small angles only from the direction of the tangent to the corresponding‡ plane trajectory at the same time. This is true of the early part of all trajectories, and for the whole of a trajectory whose initial elevation is less than 45 degrees—at any rate, when the muzzle velocity is fairly large Under these circumstances we may follow the classical§ treatment in regarding the plane trajectory as a first approximation to the actual trajectory. It is then convenient to refer the motion to axes moving with the tangent to this plane trajectory. The axis O1 is the tangent to the plane trajectory drawn in the direction of motion; axis O2 is the upward normal; and axis O3 is horizontal and to the right, as viewed from the gun. The components of $\mathbf{\Lambda}$ and $\mathbf{X}$ are (l, m, n) and (x, y, z), which are therefore the direction cosines of OA and OP respectively.

It will now be shown that it is possible to express the complete motion approximately in terms of the two complex variables, $m+in$ and $y+iz$, and the elements of the plane trajectory. We suppose that the equations of the plane trajectory have been numerically solved, so that, *e.g.*, v_1 and θ_1, the velocity and inclination in the plane trajectory, may be regarded as tabulated functions of t.

* The vector $\mathbf{G}$ may, if desired, be regarded as representing any force which acts through the centre of gravity and is a function of position only.

† See § 4.21.

‡ The *corresponding* plane trajectory is the trajectory which would be described by the same shell, with the same initial velocity and initial direction of motion, if its yaw remained always zero.

§ See, *e.g.*, CRANZ, 'Zeitschrift für Math. u. Phys.' The equations we obtain, however, appear to be new.

The components of $\boldsymbol{\Theta}$ are $(0, 0, \theta'_1)$. Using the foregoing values of the components of $\boldsymbol{\Lambda}$, $\mathbf{X}$ and $\boldsymbol{\Theta}$ in equations (3.104) and (3.105), we obtain

$$(3.201) \qquad \mathrm{N}' = -\mathrm{N}\Gamma,$$

as before; the second and third components of (3.105) give

$$(3.202) \qquad \mathrm{AN}m' + \mathrm{B}\,(nl'' - ln'') - 2\mathrm{B}nm'\theta'_1 - \mathrm{B}mn\theta''_1 + \mathrm{AN}l\theta'_1 - \mathrm{B}nl\theta'^2_1 = \mu\,(zl - xn),$$

$$(3.203) \qquad \mathrm{AN}n' + \mathrm{B}\,(lm'' - ml'') - 2\mathrm{B}nn'\theta'_1 - \mathrm{B}n^2\theta''_1 + \mathrm{B}\theta''_1 = \mu\,(xm - yl).$$

To solve the equations it is necessary to neglect certain terms. A discussion of the relative magnitude of the terms neglected, in various circumstances, will be given later in § 4.3. Some of these terms are negligible in all cases, on account of the smallness of θ'_1 in comparison with the angular spin of the shell. Others are only negligible so long as the yaw δ is so small that $1-\cos\delta$ and $1-\sin\delta/\delta$ may be neglected in comparison with unity. By such arguments it is not difficult to justify the reduction of these equations to the form

$$(3.204) \qquad \mathrm{AN}m' - \mathrm{B}n'' + \mathrm{AN}\theta'_1 = \mu\,(z - n),$$

$$(3.205) \qquad \mathrm{AN}n' + \mathrm{B}m'' + \mathrm{B}\theta''_1 = \mu\,(m - y).$$

For the particular case of the initial motion of the shells from the gun rifled 1 turn in 30 calibres in the present trials, the terms neglected are, in general, less than 1 per cent. of some term retained, and the coefficients of equations (3.204) and (3.205) may be regarded as affected by possible 1 per cent. errors. Even in the case of the gun rifled 1 turn in 40 calibres, where values of δ as great as 7 degrees or more are met with among the stable rounds, the employment of (3.204) and (3.205) is justifiable.

We now define new variables and constants by the equations

$$\eta + c\xi = m + in, \qquad c\xi = y + iz,$$
$$\mathrm{AN/B} = \Omega, \qquad c = \cos\theta_1, \qquad \theta'_1 + i\theta''_1/\Omega = \Phi.$$

If we multiply (3.204) by i, and subtract from (3.205), we obtain

$$(3.206) \qquad \frac{d^2}{dt^2}(\eta + c\xi) - i\Omega\frac{d}{dt}(\eta + c\xi) - \frac{\mu\eta}{\mathrm{B}} = i\Omega\Phi.$$

So long as the yaw remains small, equations (3.201) and (3.206) may be taken as equivalent to (3.104) and (3.105).

3.21. *The Motion of the Centre of Gravity.*—With the present axes, the components of $\mathbf{G}$ are $(-g\sin\theta_1, -g\cos\theta_1, 0)$. Equation (3.112) becomes†

$$(3.211) \qquad v' = -\mathrm{R}\,(v, \delta)/m^* - g\,(x\sin\theta_1 + y\cos\theta_1).$$

† To avoid confusion the mass of the shell is temporarily denoted by m^*

The second and third components of (3.113) become

(3.212) $$y'+x\theta_1' = \kappa(m-y\cos\delta)-(g/v)\cos\theta_1+(yg/v)(x\sin\theta_1+y\cos\theta_1),$$

(3.213) $$z' = \kappa(n-z\cos\delta)+(zg/v)(x\sin\theta_1+y\cos\theta_1).$$

The equation of the plane trajectory corresponding to (3.211) ($\delta = y = 0$, $x = 1$) is

(3.214) $$v'_1 = -\mathrm{R}(v_1, 0)/m^*-g\sin\theta_1.$$

Therefore, if $u = v-v_1$, u satisfies the equation

(3.2141) $$u' = -\{\mathrm{R}(v_1+u, \delta)-\mathrm{R}(v_1, 0)\}/m^*-g\{(x-1)\sin\theta_1+y\cos\theta_1\}.$$

In §§ 4.22, 4.31, we shall show that it is legitimate to regard the value of u determined by this equation as zero. We can therefore replace v by v_1 in (3.206), (3.212) and (3.213).

A further discussion shows that (3.212) and (3.213) can be reduced to

$$y' = \kappa(m-y)+(g/v_1)\,y\sin\theta_1,$$
$$z' = \kappa(n-z)+(g/v_1)\,z\sin\theta_1,$$

the accuracy and validity of these equations being the same as those of (3.204) and (3.205).† These equations combine to give

$$\frac{d}{dt}(c\xi) = \kappa\eta-\frac{gc\sin\theta_1}{v_1}\xi;$$

or, using the equation of the plane trajectory, $\theta'_1 = -(g/v_1)\cos\theta_1$,

(3.215) $$\xi' = \kappa\eta/c.$$

In the cases contemplated this equation is equivalent to (3.212) and (3.213). Then (3.215), (3.206) and the equations of the plane trajectory represent the required approximation to the complete equations of motion of the shell.

In order to convert (3.206) and (3.215) into linear differential equations, it is necessary to assume that μ and κ are independent of δ, and regard them as functions of v_1. This approximation involves errors no greater than the previous approximations. If Ω is treated as a variable, it must be determined by (3.201), Γ being regarded as a known function of the time. All the coefficients in (3.215) and (3.206) are then known functions of the time.

§ 3.3. *Equations of Motion of Type β.*

In the neighbourhood of the vertex of a trajectory of elevation as great as 70 degrees, the yaw, as stated in § 1.34, may reach large values. In such cases, the

† With the exception noted in § 4.22.

plane trajectory can no longer be regarded as a valid first approximation, and the only possible method is to obtain equations of motion which are suitable for direct step-by-step integration. For this purpose the following set of moving axes are most suitable, as they reduce the equations of motion of the centre of gravity to its simplest form. We take the true direction of motion OP for the axis 1 and a horizontal line at right angles to OP for the axis 3. We define the position of OP by spherical polar co-ordinates θ, ψ with respect to axes fixed in direction at O, see fig. 10. Then $\mathbf{X}$ has components (1, 0, 0), $\boldsymbol{\Theta}$ has components $(-\psi' \sin\theta, -\psi' \cos\theta, \theta')$, $\mathbf{G}$ has components $(-g \sin\theta, -g\cos\theta, 0)$ and $\boldsymbol{\Lambda}$ components (l, m, n) as before.

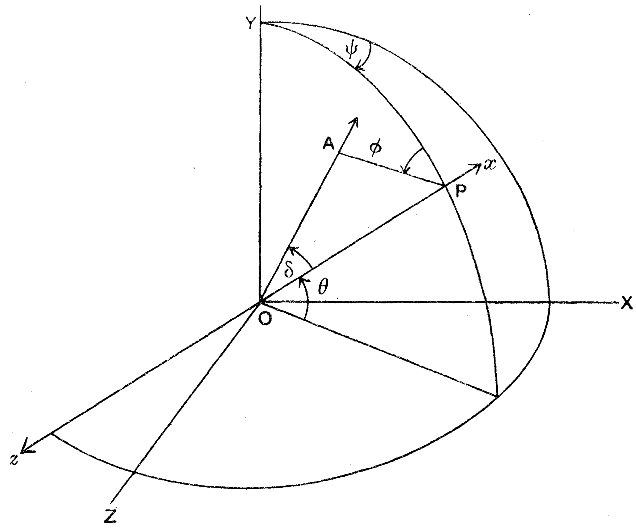


Fig. 10. OX, Y, Z are fixed axes, OY being the upwards vertical; the plane XOY contains the line of fire.

Equation (3.105), when written out in full, becomes very complicated. To simplify it, we can, under certain circumstances, neglect the angular momentum about a transverse axis compared to the angular momentum about the axis of the shell. The legitimacy of this approximation, which is equivalent to putting B = 0 in (3.105), is discussed in § 4.33. It should be stated that this type of approximation also is classical,* but that the equations we obtain are apparently new and of a wide range of validity.

As before, we have

$$N' = -N\Gamma. \tag{3.301}$$

The second and third components of (3.105) reduce to

$$\mathrm{AN}m' - \mathrm{AN}\,(-n\psi' \sin\theta - l\theta') = -\mu n,$$
$$\mathrm{AN}n' - \mathrm{AN}\,(-l\psi' \cos\theta + m\psi' \sin\theta) = \mu m,$$

* See, *e.g.*, CHARBONNIER, 'Traité de Balistique Extérieure,' Livre V., Chap. IV.

or, writing ω for μ/AN,

$$m' = -n(\omega+\psi'\sin\theta)-l\theta', \tag{3.302}$$

$$n' = m(\omega+\psi'\sin\theta)-l\psi'\cos\theta. \tag{3.303}$$

The corresponding equations of motion of the centre of gravity are

$$v' = -\mathrm{R}(v,\ \delta)/m^{*}-g\sin\theta, \tag{3.304}$$

$$\theta' = \kappa m-(g/v)\cos\theta, \tag{3.305}$$

$$\psi'\cos\theta = \kappa n. \tag{3.306}$$

The six equations (3.301) to (3.306) can be solved by a step-by-step process, if R, κ, μ and Γ are numerically known functions of v and δ. They are valid without restriction as to the size of δ, and have proved of value for the discussion of trajectories at very high elevations. They are, however, necessarily invalid when any question of stability is under discussion.

§ 3.4. *Equations of Motion of Type* γ.

For the purpose of discussing the initial motion of a shell which is unstable or just stable, equations of types α and β are invalid, and it is necessary to make use of equations corresponding to the equations of energy and angular momentum for a top. The equations we shall thus obtain are of far less general applicability than types α and β.

With this object we take the scalar product of both sides of equation (3.105) into the vector $[\mathbf{\Lambda}\,.\,\mathbf{\Lambda}']+\mathbf{\Theta}$, and obtain, after reduction,

$$\tfrac{1}{2}\mathrm{B}\frac{d}{dt}\left\{(\mathbf{\Lambda}')^2+2(\mathbf{\Theta}\,.\,[\mathbf{\Lambda}\,.\,\mathbf{\Lambda}'])+(\mathbf{\Theta})^2-(\mathbf{\Lambda}\,.\,\mathbf{\Theta})^2\right\} = -\mu(\mathbf{X}\,.\,\{\mathbf{\Lambda}'-[\mathbf{\Lambda}\,.\,\mathbf{\Theta}]\}). \tag{3.401}$$

Using the axes described in the last section, we note that, over a limited range at the beginning of a trajectory, the first two components of $\mathbf{\Theta}$ are numerically very small compared to the third, θ'. We shall find that the effect of θ' itself is negligible in the cases we consider. We shall therefore neglect the other components of $\mathbf{\Theta}$ at once. Taking δ and ϕ as spherical polar co-ordinates of the axis OA referred to the moving axes, so that

$$l = \cos\delta,\ m = \sin\delta\cos\phi,\ n = \sin\delta\sin\phi,\dagger$$

† The angle ϕ is not exactly the angle measured by the jump cards, but the difference is negligible. The angle δ is exactly the measured angle of yaw.

we find that (3.401) reduces to

$$(3.402)\quad \tfrac{1}{2}\mathrm{B}\frac{d}{dt}\left\{\delta'^2+\phi'^2\sin^2\delta+2\theta'(\delta'\cos\phi-\phi'\sin\delta\cos\delta\sin\phi)+\theta'^2(1-\sin^2\delta\sin^2\phi)\right\}$$
$$= -\mu\left\{\frac{d\cos\delta}{dt}-\theta'\sin\delta\cos\phi\right\}.$$

This is the equation corresponding to the equation of energy. The first component of (3.105) corresponds to the equation of angular momentum for a top, and reduces in the same way to

$$(3.403)\quad \frac{d}{dt}\{\mathrm{AN}\cos\delta+\mathrm{B}\phi'\sin^2\delta\}+2\mathrm{B}\theta'\delta'\sin^2\delta\sin\phi-\mathrm{B}\theta''\sin\delta\cos\delta\sin\phi$$
$$-\mathrm{AN}\theta'\sin\delta\cos\phi+\mathrm{B}\theta'^2\sin^2\delta\cos\phi\sin\phi = 0.$$

Equation (3.201) remains unaltered. Over the range of the jump card experiments a mean value of δ'/θ' is 50. We shall therefore regard it as legitimate for our present purposes to neglect all terms containing θ'. On integrating the resulting equations we obtain

$$(3.404)\qquad \tfrac{1}{2}\mathrm{B}(\delta'^2+\phi'^2\sin^2\delta)+\int_0^\delta \mu d\cos\delta = \tfrac{1}{2}\mathrm{BE},$$

$$(3.405)\qquad \mathrm{AN}\cos\delta+\mathrm{B}\phi'\sin^2\delta = \mathrm{BF},$$

where E and F are constants of integration. In (3.405) it is assumed that N is constant. If μ is constant these equations are of the same form as those of the motion of a top. In the more important applications to the jump card trial which we shall make of (3.404) and (3.405), μ will be treated as a variable function of δ, and also of v.

§ 3.5. *The Additional Force Components* H, J *and* K.

It is now necessary to consider the effect of the additional force components, mentioned in §§ 1.1, 1.12, and denoted by H, J and K. These have so far been omitted from the general equations for the sake of simplicity. The couples H and J will affect the angular motion of the axis, and the force K will affect the motion of the centre of gravity. For algebraic convenience we define new variables h, γ, λ, by the equations

$$\mathrm{H} = h\mathrm{B}w,\qquad \mathrm{J} = \mathrm{AN}\gamma\sin\delta,\qquad \mathrm{K} = m\mathrm{N}v\lambda\sin\delta,$$

where w is the total angular velocity of the axis of the shell. The force components may then be represented, in the notation of § 3.1, by the following vectors:—

H by the vector $-h\mathrm{B}\{[\mathbf{\Lambda}\,.\,\mathbf{\Lambda}']-(\mathbf{\Lambda}\,.\,\mathbf{\Theta})\,\mathbf{\Lambda}+\mathbf{\Theta}\}$;

J by the vector $\mathrm{AN}\gamma(\mathbf{\Lambda}\cos\delta-\mathbf{X})$;

K by the vector $m\mathrm{N}v\lambda[\mathbf{\Lambda}\,.\,\mathbf{X}]$.

To include the effect of these components we add to the right-hand side of (3.105)

$$-h\mathrm{B}\{[\boldsymbol{\Lambda}\,.\,\boldsymbol{\Lambda}']-(\boldsymbol{\Lambda}\,.\,\boldsymbol{\Theta})\,\boldsymbol{\Lambda}+\boldsymbol{\Theta}\}+\mathrm{AN}\gamma\,(\boldsymbol{\Lambda}\cos\delta-\mathbf{X}),$$

and to the right right-hand side of (3.113)

$$\mathrm{N}\lambda\,[\boldsymbol{\Lambda}\,.\,\mathbf{X}].$$

Equations (3.104) and (3.112) (type α) are unaltered. As a result, the following additions must be made to the right-hand side of succeeding equations :—

To (3.202),	$+h\mathrm{B}n'+\mathrm{AN}\gamma\,(m-y)$.
To (3.203),	$-h\mathrm{B}m'+\mathrm{AN}\gamma\,(n-z)$.
To (3.212),	$-\mathrm{N}\lambda\,(z-n)$.
To (3.213),	$-\mathrm{N}\lambda\,(m-y)$.

As the total effect of the extra components h, γ and λ is certainly small in any practical case, we have neglected all terms other than those of the lowest order in δ. Equations (3.206) (3.215), when modified by the inclusion of these extra terms, become

$$(3.501)\qquad \frac{d^2}{dt^2}(\eta+c\zeta)-(i\Omega-h)\frac{d}{dt}(\eta+c\zeta)-\left(\frac{\mu}{\mathrm{B}}-i\Omega\gamma\right)\eta=i\Omega\Phi\,;$$

$$(3.502)\qquad \zeta'=(\kappa-i\mathrm{N}\lambda)\,\eta/c.$$

3.51. *The Additional Terms in Equations of Types β and γ.*—The additional terms in the equations of type β can be written down in a similar manner. The following additions must be made to the right-hand sides of the equations :—

To (3.302),	$+\gamma m\cos\delta-h\,(nl'-ln')/\Omega$.
To (3.303),	$+\gamma n\cos\delta-h\,(lm'-ml')/\Omega$.
To (3.305),	$+\mathrm{N}\lambda n$.
To (3.306),	$-\mathrm{N}\lambda m$.

The terms in h are negligible, as they are $\mathrm{O}\,(h\delta'/\Omega^2)$ compared with the principal terms $-n\omega-l\theta'$, so long as ω/Ω is not very small. The principal application of these equations is to the motion of a shell near the vertex of a trajectory at an elevation of 70 degrees, where the velocity becomes small while the spin probably remains large. Under these circumstances the terms γ and λ arising from the spin rise in importance relatively to the terms ω and κ representing the ordinary force components. The inclusion of the extra terms γ and λ in these equations is at present of no practical importance, as we have no definite information as to their value.

The corresponding terms could be added to equations of type γ by the same methods, but the results are of no importance as it is impossible at present to solve these equations unless these terms are neglected.

§ 3.6. *The Approximate Solution of Equations of Type* α.

3.61. *The Nature of the Solution Required.*—The system of equations (3.501), (3.502) are linear differential equations with respect to the time of the second order, the coefficients being regarded as known functions of the time t. Since these functions are in practice empirical and by no means simple, an exact solution is impossible. To simplify the discussion we write

$$s = A^2N^2/4B\mu, \qquad \mu/B = \Omega^2/4s,$$

so that equations (3.501), (3.502) become

$$\frac{d^2}{dt^2}(\eta+c\zeta)-(i\Omega-h)\frac{d}{dt}(\eta+c\zeta)-\left(\frac{\Omega^2}{4s}-i\Omega\gamma\right)\eta = i\Omega\Phi\,; \tag{3.611}$$

$$\zeta'-(\kappa-i\mathrm{N}\lambda)\,\eta/c = 0. \tag{3.612}$$

If the terms in ζ, h, γ be omitted from (3.611), and s, N and Ω are assumed constant, the equation reduces to that for the small oscillations of a top in the neighbourhood of the vertical.

The coefficient s is the stability factor as defined in § 1.31. In order to be able to apply the approximations on which (3.611) and (3.612) are based, we shall find that it is necessary to assume that the shell is more than just stable, *e.g.*, $s > 1\cdot 1$.

We proceed to develop an approximate solution of the equations on the assumption that Ω is large. If we ignore the dimensions of the various terms, and take the unit of time as 1 second, then Ω is in practice greater than 100 (radians per second), all other terms being of the order unity. This is really equivalent to assuming that all the ratios κ/Ω, h/Ω, ..., which are of no dimensions, are small. It will be found necessary to assume further that all derivatives with respect to the time are of order unity in units of 1 second, *e.g.*, that κ', κ'', s', Ω'..., are of order unity. These conditions are satisfied in practice. As a result, we can say that κ/Ω, s'/Ω..., are small quantities of the first order, and κ'/Ω^2, s''/Ω^2, θ''_1/Ω^2..., are small quantities of the second order. For simplicity, we shall throughout ignore dimensions, and denote such terms of the first order by $O(1/\Omega)$, and terms of the second order by $O(1/\Omega^2)$.* The arithmetical values of the various terms are investigated in detail in § 4.3 below.

The above facts indicate the lines on which an approximate solution is to be sought—*we require the asymptotic expansion of the solution (or its leading terms) for large values of the parameter* Ω. Methods of obtaining such expansions have

* In practice the spin N, and therefore Ω, decreases slightly along the trajectory, but the diminution is not sufficient to affect the assumption that Ω is large.

been investigated in general terms by HORN and SCHLESINGER.* A method, which is slightly different algebraically, is more convenient here; the asymptotic properties of our solutions, however, may be regarded as established by the researches of these authors.

The equations (3.611) and (3.612) are a pair of linear differential equations with respect to the time for the two dependent variables η and ζ, (3.611) being of the second order and (3.612) of the first. There must, therefore, be three independent solutions.

It is convenient to eliminate ζ' and ζ'' from (3.611) by the use of (3.612), the result being

$$(3.613) \qquad \eta''-(i\Omega-h-\kappa_1)\,\eta'-\{\Omega^2/4s+i\Omega\,(\kappa_1-\gamma)-h\kappa_1-\kappa'_1-\kappa_1 c'/c\}\,\eta$$
$$-\{i\Omega c'-hc'-c''\}\,\zeta = i\Omega\Phi,$$

where κ_1 is written for $(\kappa-i\text{N}\lambda)$. It is believed that $\text{N}\lambda$ is small compared with κ, so that for simplicity the term $\text{N}\lambda$ will usually be omitted in subsequent work. The term γ will however be retained.

3.62. *The Complementary Function.*—A first approximation to the three independent complementary functions is obtained, following HORN and SCHLESINGER, by making the substitution,

$$\eta = e^{i\Omega x}\bar{\eta}, \qquad \zeta = e^{i\Omega x}\bar{\zeta},$$

and treating $\bar{\eta}$ and $\bar{\zeta}$ as constants in determining η' and ζ'. We also neglect all but the highest order terms in Ω in each equation. The equations then reduce to

$$(3.621) \qquad (-\Omega^2 x'^2+\Omega^2 x'-\Omega^2/4s)\,\bar{\eta}-i\Omega c'\bar{\zeta} = 0\,;$$

$$(3.622) \qquad -\kappa\bar{\eta}/c+i\Omega x'\bar{\zeta} = 0.$$

On eliminating $\bar{\eta}$ and $\bar{\zeta}$, and retaining only the terms of highest order in Ω, these reduce to

$$x'\,(x'^2-x'+1/4s) = 0,$$

a cubic equation for x' whose three roots correspond to the three independent

* J. HORN, 'Mathematische Annalen,' vol. 52, p. 271 and p. 340. L. SCHLESINGER, *ibid.*, vol. 63, p. 277; 'Comp. Rend.,' vol. 142, p. 1031. The investigations of the complementary function given by these writers are fairly complete, the asymptotic nature of the expansions being established. The latter writer considers a system of n linear differential equations. A similar treatment of the complementary function and the particular integral of a special equation is suggested (without proof) by M. DE SPARRE 'Atti (Rendiconti) della R. Acc. dei Lincei,' 1898, Ser. V., vol. 72, p. 111; this writer was obviously led to the solution he gives by his researches on the motion of spinning projectiles.

[*Note added July* 30, 1920. See also G. D. BIRKHOFF, 'Trans. Amer. Math. Soc', vol. 9, p. 219.]

solutions required. The roots are $\frac{1}{2}\pm\frac{1}{2}(1-1/s)^{\frac{1}{2}}$, 0, or writing, for shortness, $\sigma = (1-1/s)^{\frac{1}{2}}$, the three values of x are

$$x_1 = \tfrac{1}{2}\int_0^t (1+\sigma)\,dt\,; \qquad x_2 = \tfrac{1}{2}\int_0^t (1-\sigma)\,dt\,; \qquad x_3 = 0.$$

It appears that the first two solutions correspond to the complementary function of equation (3.613) with the term in ζ neglected, so that η is large compared with ζ. If $s < 1$, σ is imaginary, the motion is unstable and the solution fails. In the third solution ζ is large compared with η, and a first approximation to it gives a constant value to ζ, obtained by neglecting the term in η in equation (3.612). It is convenient to obtain the first two solutions independently by a special method.

We first omit the term in ζ in equation (3.613); it is not required till the second approximation. Write the equation, for simplicity, in the form

(3.623) $$\eta''-iA\eta'-B\eta = 0,$$

where

$$A = \Omega+ih+i\kappa,$$
$$B = \Omega^2/4s+i\Omega(\kappa-\gamma)-h\kappa-\kappa'-\kappa c'/c.$$

Remove the second term by substituting

$$\eta = y\exp\left\{\tfrac{1}{2}i\int_0^t A\,dt\right\},$$

giving

(3.6231) $$y''+My = 0,$$

where

$$M = \tfrac{1}{4}A^2-B+\tfrac{1}{2}iA'$$
$$= \tfrac{1}{4}\Omega^2\sigma^2\left\{1+\frac{2i}{\Omega\sigma^2}(h-\kappa+2\gamma+N'/N)+O\left(\frac{1}{\Omega^2}\right)\right\}.$$

Substitute $y = Re^{\pm iP}$, so that (3.6231) becomes

(3.6232) $$R''\pm(2iP'R'+iP''R)-P'^2R+MR = 0.$$

We may make P and R satisfy any single relation we choose, *e.g.*,

$$2P'R'+P''R = 0,$$

giving $P' = 1/R^2$,* so that (3.6232) becomes

(3.6233) $$R''-1/R^3+MR = 0.$$

* More generally $P' = a/R^2$, where a is a constant, but the value of this constant is immaterial, as it disappears in the result.

This equation may be solved asymptotically, by successive approximation, by writing*

$$R = R_0(1+R_1+R_2\ldots),$$

where

$$R_1 = O(1/M) = O(1/\Omega^2), \qquad R_2 = O(1/\Omega^4)\ldots.$$

We obtain, in succession, the approximations

$$R_0 = M^{-\frac{1}{4}},$$

$$R_1 = -\tfrac{1}{4}M^{-\frac{3}{4}}\frac{d^2}{dt^2}(M^{-\frac{1}{4}}),$$

verifying the relation $R_1 = O(1/\Omega^2)$. The order of magnitude of R_1 in practice will be discussed in § 4.32, where it will be shown to be negligible.† We therefore take as our two standard solutions

$$y_1 = M^{-\frac{1}{4}}\exp\left\{i\int_0^t M^{\frac{1}{2}}\,dt\right\},$$

$$y_2 = M^{-\frac{1}{4}}\exp\left\{-i\int_0^t M^{\frac{1}{2}}\,dt\right\},$$

giving, for the complementary function of (3.623),

(3.6234) $$\eta = (\Omega\sigma)^{-\frac{1}{2}}\{1+O(1/\Omega)\}\{K_1e^{iP_1}+K_2e^{iP_2}\},$$

where K_1, K_2 are arbitrary constants, and P_1, P_2 are given by

$$P_1,\ P_2 = \tfrac{1}{2}\int_0^t [\Omega(1\pm\sigma)+i\{h+\kappa\pm(h-\kappa+2\gamma+N'/N)/\sigma\}]\,dt.$$

This is the form of solution which is used in analysing the jump card experiments, and contains all the terms that can be required in practice.

It is now necessary to examine the effect of the term in ζ in (3.613), which has so far been omitted. The value of ζ', obtained from (3.612), corresponding to the first solution for η, is

$$\zeta' = (\kappa/c)(\Omega\sigma)^{-\frac{1}{2}}e^{iP_1},$$

so that, on integrating by parts to obtain the leading terms,

$$\zeta_1 = \frac{2\kappa\eta_1}{ic\Omega(1+\sigma)} + O\left(\frac{1}{\Omega^2}\right).$$

* At this point the advantage of our *ad hoc* method over more general methods is apparent, as we obtain in one step a solution with an error $O(1/\Omega^2)$, whereas the general method requires two steps.

† We assume that the numerical value of R_1, the next term in the expansion, is a measure of the error in the solution caused by omitting all terms after the first. The expansions for y_1 and y_2 are known to be asymptotic for large values of Ω, so that the error will be some finite multiple of R_1, but the size of the numerical factor is unknown.

Similarly we have

$$\zeta_2 = \frac{2\kappa\eta_2}{ic\Omega(1-\sigma)} + O\left(\frac{1}{\Omega^2}\right),$$

verifying that ζ is small compared to η. The contribution of ζ to equation (3.613) is thus

(3.6235) $$-\frac{2\kappa c'\eta}{c(1\pm\sigma)} + O\left(\frac{1}{\Omega}\right),$$

which is equivalent to an addition to the coefficient of η of terms which are $O(1/\Omega^2)$ compared to the principal term. The solution can be repeated with these terms included, but is unaffected to the order to which we are working. We shall take our first two standard solutions in the form

(3.624) $$\eta_1 = \left(\frac{\Omega_0\sigma_0}{\Omega\sigma}\right)^{\frac{1}{2}} e^{iP_1}, \qquad \zeta_1 = \frac{2\kappa\eta_1}{ic\Omega(1+\sigma)},$$

(3.625) $$\eta_2 = \left(\frac{\Omega_0\sigma_0}{\Omega\sigma}\right)^{\frac{1}{2}} e^{iP_2}, \qquad \zeta_2 = \frac{2\kappa\eta_2}{ic\Omega(1-\sigma)}.$$

The differential coefficients of the solutions may be obtained by differentiation of these equations.

For the third solution we have shown that the exponential index is zero to the first order, and that a first approximation is given by

$$\eta_3 = \zeta'_3 = 0, \qquad \zeta_3 = 1.$$

The expansions take a somewhat different form, like those for the particular integral, and we write

(3.6261) $$\eta_3 = \eta^{(0)} + \eta^{(1)}/i\Omega + \eta^{(2)}/(i\Omega)^2 \ldots,$$

(3.6262) $$\zeta_3 = \zeta^{(0)} + \zeta^{(1)}/i\Omega + \zeta^{(2)}/(i\Omega)^2 \ldots.$$

Substituting in equations (3.612) and (3.613), we obtain

$$\eta^{(1)} = 4sc' = -4s\,\theta'_1 \sin\theta_1,$$

$$\zeta^{(1)} = 4\int_0^t \kappa sc'\,dt/c.$$

The significance of this solution will be considered after the particular integral has been discussed.

Our standard third solution is then

(3.627) $$\eta_3 = \frac{4sc'}{i\Omega}, \qquad \zeta_3 = 1 + \frac{4}{i\Omega}\int_0^t \frac{\kappa sc'}{c}\,dt.$$

3.63. *The Particular Integral.*—The question of the particular integral is not treated generally by HORN and SCHLESINGER. The former considers shortly a very particular case.* Their methods can, however, be extended to obtain the results we require.

We assume an expansion for the particular integral $\bar{\eta}$, $\bar{\zeta}$ of the form of (3.6261), (3.6262). This integral can be specified in such a way that initially $\bar{\zeta} = 0$, *i.e.*, $\zeta^{(0)} = \zeta^{(1)} = \ldots = 0$, and $\eta^{(0)} = 0$. It will then be found to be unique.

On substituting in equations (3.612) and (3.613), with the right-hand side retained, and equating powers of $i\Omega$, we find first that $\zeta^{(0)} = \eta^{(0)} = 0$ for all time, and then

$$\eta^{(1)} = 4s\Phi, \qquad \zeta^{(1)} = \int_0^t 4s\kappa\Phi dt/c. \tag{3.631}$$

The first two terms in the expansions of $\bar{\eta}$ and $\bar{\zeta}$ take the forms

$$\bar{\eta} = \frac{4s\Phi}{i\Omega} + \frac{4s}{(i\Omega)^2}\left\{\frac{d}{dt}\ (4s\Phi) + (\kappa - \gamma)\,4s\Phi + c'\int_0^t 4s\kappa\Phi dt/c\right\}; \tag{3.632}$$

$$\bar{\zeta} = \int_0^t \kappa\bar{\eta} dt/c. \tag{3.633}$$

Equations (3.632) and (3.633) will be taken as the standard particular integral. Since, moreover, they contain no periodic terms, and the initial value of $\bar{\zeta}$ is zero and those of $\bar{\eta}$ and $\bar{\eta}'$ very small in practice, it is convenient to take this solution as the standard solution of the equations of motion in cases where the initial values of η and η' are not exactly known—*e.g.*, in calculating the drift.

The expansions for $\bar{\eta}$ and $\bar{\zeta}$, of which the first two terms are given above, can be shown to be asymptotic, but we cannot take up this question here. The numerical accuracy of (3.632) and (3.633) will be considered in § 4.33.

3.64. The general solution of (3.612) and (3.613) may be put in the form

$$\eta = K_1\eta_1 + K_2\eta_2 + K_3\eta_3 + \bar{\eta}, \tag{3.641}$$

$$\zeta = K_1\zeta_1 + K_2\zeta_2 + K_3\zeta_3 + \bar{\zeta}, \tag{3.642}$$

where K_1, K_2, K_3 are arbitrary complex constants and $\eta_1, \ldots, \zeta_1, \ldots$, have the values determined in the last section.

The particular integral $\bar{\eta}$, $\bar{\zeta}$ represents the motion in an actual trajectory in which ζ is initially zero, and η and η' start with what may be called their equilibrium values, which are numerically very small. The solution $(K_3\eta_3 + \bar{\eta})$, $(K_3\zeta_3 + \bar{\zeta})$ represents the motion, of the same type, in a trajectory whose initial tangent makes an angle (determined by K_3) with the initial tangent of the chosen plane trajectory. This can

* *Loc. cit.*, p. 340. We hope to publish these extensions in another place.

be seen from the following considerations. The motion in a slightly different *plane trajectory* would be obtained by omitting all terms in η from the equations of motion of the centre of gravity, and ignoring the equations of angular motion. Equation (3.612) then reduces to $\zeta' = 0$; this represents a trajectory which only differs from a varied *plane trajectory* on account of terms omitted in § 3.21, whose retention renders the equation non-linear. The value of η_3 in (3.627) gives the alteration, through the change in direction of projection, of the first term in $\bar{\eta}$.

3.65. In the usual practical case, the initial conditions take the form

$$\zeta_0 = 0, \qquad \eta_0 = a, \qquad \eta'_0 = b\Omega,$$

where a and b are arbitrary complex constants. It is desirable in such a case to know the degree of importance of the three standard solutions.

The initial values of the standard solutions (retaining the highest order terms only) are as follows:—

$$\begin{array}{lll} \eta_1 = 1, & \zeta_1 = O(1/\Omega), & \eta'_1 = \tfrac{1}{2}i\Omega(1+\sigma), \\ \eta_2 = 1, & \zeta_2 = O(1/\Omega), & \eta'_2 = \tfrac{1}{2}i\Omega(1-\sigma), \\ \eta_3 = O(1/\Omega), & \zeta_3 = 1, & \eta'_3 = O(1/\Omega), \\ \bar{\eta} = O(1/\Omega), & \bar{\zeta} = 0, & \bar{\eta}' = O(1/\Omega). \end{array}$$

The constants K_1, K_2, K_3 are determined by the equations

$$K_1\eta_1 + K_2\eta_2 + K_3\eta_3 + \bar{\eta} = a,$$

$$K_1\eta_{1'} + K_2\eta_{2'} + K_3\eta_{3'} + \bar{\eta}' = b\Omega,$$

(3.6501) $$K_1\zeta_1 + K_2\zeta_2 + K_3\zeta_3 + \bar{\zeta} = 0.$$

Retaining only the highest order terms these reduce to

(3.651) $$K_1 + K_2 = a,$$

(3.652) $$\tfrac{1}{2}i(1+\sigma)K_1 + \tfrac{1}{2}i(1-\sigma)\kappa_2 = b,$$

(3.653) $$K_3 + O(1/\Omega) = 0.$$

It follows at once that $K_3\eta_3$ is completely negligible compared to $K_1\eta_1$ and $K_2\eta_2$, and that in investigating η we may ignore the third solution (and the particular integral) altogether. On the other hand, the contributions of all the solutions to ζ are of the same order of importance. We shall therefore take as the solution satisfying the most general initial conditions—

(3.654) $$\eta = K_1\eta_1 + K_2\eta_2,$$

where K_1 and K_2 are determined by (3.651) and (3.652), and

(3.655) $$\zeta = K_1\zeta_1 + K_2\zeta_2 + K_3\zeta_3 + \int_0^t \kappa\bar{\eta}\,dt/c,$$

where by (3.6501),

(3.656) $$K_3 = -K_1(\zeta_1)_0 - K_2(\zeta_2)_0.$$

§ 3.7. *The Solution of Equations of Type* γ.

The equations of type β are only soluble numerically by step-by-step integration, and will not be considered here, but the equations of type γ (§ 3.4) reduce, when μ is constant and damping effects are neglected, to the equations of a spinning top, and it is convenient to summarize here their solution, in terms of elliptic functions, in the form which is most suitable for our purposes. We shall only consider the initial conditions $\delta = 0$, $\delta' = b\Omega$; this is the rosette form of motion (§ 1.3) and is usually a good approximation to the true motion in its earliest stage. In this case we obtain from (3.404) and (3.405)

(3.701) $$\phi' = \Omega/(1+\cos\delta),$$

(3.702) $$\delta'^2\sin^2\delta - \Omega^2 b^2\sin^2\delta + \Omega^2(1-\cos\delta)^2 - (\Omega^2/2s)(1-\cos\delta)\sin^2\delta = 0.$$

If we take Ωt as independent variable, the motion depends only on two parameters, b and s. The solution of (3.702) is given by

(3.703) $$\sin\tfrac{1}{2}\delta = \sin\tfrac{1}{2}\alpha\,\mathrm{cn}\,(K-\lambda\Omega t,\ k),$$

where α, λ, and k are given by the formulæ

(3.7041) $$\sqrt{s} = \cos\tfrac{1}{2}\alpha\cosh\tfrac{1}{2}c,$$

(3.7042) $$b = \tan\tfrac{1}{2}\alpha\tanh\tfrac{1}{2}c,$$

(3.7043) $$\tan\epsilon = \sin\tfrac{1}{2}\alpha/\sinh\tfrac{1}{2}c, \qquad (k = \sin\epsilon),$$

(3.7044) $$\lambda = (\sin\tfrac{1}{2}\alpha)/2k\sqrt{s},$$

and K is the complete elliptic integral of the first kind to modulus k. Thus the yaw oscillates between the values 0 and α, and the value of the period T—the interval between successive zeros—is given by

(3.705) $$\Omega T = 2K/\lambda.$$

The curve of yaw, δ, plotted against Ωt is initially concave (convex) upwards, when $s < 1\,(>1)$. This corresponds to the case of instability (stability) for small oscillations.

In the practical analysis of the results of rounds whose stability factor is less than or near 1, it is convenient to use graphical methods. If the observed yaw is plotted against Ωt, it is easy to read off the observed values of α, the maximum yaw, and ΩT, the period. A chart was therefore constructed with suitable families of curves, according to (3.7041)–(3.7044), from which, when ΩT and α are known, s, b, c, and k can be read off directly.

PART IV.—ANALYSIS OF THE EXPERIMENTAL RESULTS.

§ 4.0. *Equations of Motion in Polar Co-ordinates.*

The theoretical results of Part III. will now be applied to the analysis of the observations described in Part II., which consist of determinations of yaw δ and orientation of yaw ϕ, for a shell fired horizontally over a range of about 600 feet. When the stability factor is greater than about 1·1, the maximum yaw for the corresponding round never exceeds 7 degrees, and it is then possible to make use of the complementary function solution of equations of type α as given in § 3.6. These rounds give more valuable information than those which are less stable.

We treat certain of the force coefficients as constants over the range of the experiments, and verify that the results of the theory agree with experiment when certain values are given to the force coefficients. In particular the spin is treated as constant. The way in which the coefficients vary with the velocity is determined mainly by firing shells with various muzzle velocities. The final results have been already described in § 1.2 above.

The experiments determine the values, at definite time intervals along the range (§ 2.0), of the angle of yaw δ and the angle ϕ turned through by the line in which the plane of yaw meets the cards. The measured value of ϕ is zero, when this line is vertical and increases from 0 to 2π radians in the direction in which the shell is spinning. It is, of course, ambiguous by an integral multiple of 2π. Except where specially stated the yaw δ is assumed to be an essentially positive quantity. When OA passes through the position OP, the yaw vanishes; the value of ϕ will change discontinuously by an amount $\pm\pi$, and $d\delta/dt$ will change its sign discontinuously.

It is convenient in Part IV. to express the solution of the equations of motion of type α in terms of the co-ordinates δ and ϕ. The exact relations between the measured δ and ϕ and the direction cosines (l, m, n) and (x, y, z) of § 3.2 are

$$\cos\delta = lx+my+nz,$$

$$\tan\phi = \frac{(nx-lz)\cos\theta_1-(ny-mz)\sin\theta_1}{mx-ly},$$

where θ_1 is the inclination to the horizontal of the tangent to the plane trajectory. Since $\theta_1 < 1\frac{1}{2}$ degrees, we may replace the latter by

$$\tan\phi = (nx-lz)/(mx-ly).$$

Since η is defined by the equation

$$\eta = (m-y)+i(n-z),$$

we obtain, when δ is sufficiently small,

$$\eta = \sin\delta e^{i\phi};$$

this expression neglects terms of the second order compared to those retained. It is an adequate approximation provided $\delta < 7$ degrees.

The general solution for the equations of type α, given in § 3.65, equations (3.654) and (3.6234), is*

(4.01) $$\eta = (\sigma_0/\sigma)^{\frac{1}{2}}(\mathrm{K}_1 e^{i\mathrm{P}_1}+\mathrm{K}_2 e^{i\mathrm{P}_2}),\dagger$$

if we ignore, as we may, the particular integral and the third solution. We shall write

$$\mathrm{P}_1 = p_1+iq_1+p_2+iq_2,$$
$$\mathrm{P}_2 = p_1+iq_1-(p_2+iq_2).$$

Then

(4.011) $$p_1 = \tfrac{1}{2}\int_0^t \Omega\,dt = \tfrac{1}{2}\Omega t, \qquad p_2 = \tfrac{1}{2}\int_0^t \Omega\sigma\,dt,$$

(4.012) $$q_1 = \tfrac{1}{2}\int_0^t (h+\kappa)\,dt, \qquad q_2 = \tfrac{1}{2}\int_0^t (h-\kappa+2\gamma)\,dt/\sigma,$$

and $\sigma^2 = 1-1/s$. We observe that p_1, p_2, q_1, q_2 are all nearly proportional to the time t.

The general solution (equation (4.01)) contains two complex arbitrary constants or four real constants. By a suitable choice of origin for t and ϕ these may be reduced to two. If the time $t = 0$ corresponds to a minimum of δ and the value $\phi = 0$, equation (4.01) may be written

(4.02) $$\eta = \mathrm{J}(\sigma_0/\sigma)^{\frac{1}{2}} e^{ip_1-q_1}\{\cos p_2 \sinh(j-q_2)+i\sin p_2\cosh(j-q_2)\},$$

* Treating N and Ω as constant, *i.e.*, neglecting the spin reducing couple Γ.

† Equation (4.01) reduces approximately to the form $\eta = \mathrm{K}t$, when $s = 1$, and to the form $\eta = (\sigma_0/\sigma)^{\frac{1}{2}}\{\mathrm{K}_1 e^{\phi_1}+\mathrm{K}_2 e^{\phi_2}\}$, when $s < 1$, and the shell unstable, the principal parts of ϕ_1, ϕ_2 being real and positive. The solution then fails completely as an approximation to the actual motion except over a small part of the first period. As s approaches the value unity from above, the errors from this cause will begin to increase, but the magnitude of these errors can only be estimated by comparison with the solution of equations of type γ, see § 4.3 below.

where J and j are new arbitrary constants; of these j is small if $|\eta|$ is small at $t = 0$.* The motion is a combination of the following components:—

(1) A uniform rotation about the origin, represented by the term e^{ip_1}.

(2) A damping of the amplitude, represented by $(\sigma_0/\sigma)^{\frac{1}{2}} e^{-q_1}$.

(3) An oscillation of period determined by p_2 whose phase is continually changed by the factor $(j-q_2)$. The values of δ and ϕ are given by the equations

(4.031) $$\delta^2 = \tfrac{1}{2}J^2 (\sigma_0/\sigma) e^{-2q_1} \{\cosh 2(j-q_2) - \cos 2p_2\},$$

(4.032) $$\phi = \phi_0 + p_1 + \text{arc tan} \{\coth (j-q_2) \tan p_2\}.\dagger$$

So long as $(j-q_2)$ does not change sign, the average rate of increase of ϕ over any number of complete periods is $(p'_1 \pm p'_2)$.

Let α and β be the successive maximum and minimum values of δ (assumed positive). In determining the values of α, β, and the corresponding values of t, it is legitimate to neglect the changes of q_1, q_2, and σ, which are very small in a single period p_2. The maxima and minima are then given by putting $\cos 2p_2$ equal to -1 and $+1$ respectively in (4.031). Writing

(4.041) $$\alpha_1 = J (\sigma_0/\sigma)^{\frac{1}{2}} e^{-q_1} \cosh (j-q_2),$$

(4.042) $$\beta_1 = J (\sigma_0/\sigma)^{\frac{1}{2}} e^{-q_1} \sinh (j-q_2),$$

so that α_1, β_1 are defined for all values of t, we have

(4.051) $$\alpha = \alpha_1 (T_n),$$

for values of T_n given by

(4.052) $$p_2 (T_n) = \tfrac{1}{2}(2n+1)\pi,$$

(4.053) $$\beta = |\beta_1 (T'_n)|,$$

for values of T'_n given by

(4.054) $$p_2 (T'_n) = n\pi.$$

An alternative expression for ϕ is then

(4.06) $$\phi = \phi_0 + p_1 + \text{arc tan} \left(\frac{\alpha_1}{\beta_1} \tan p_2\right).$$

* The curves of δ against t appear to have a minimum very near the muzzle of the gun in all rounds fired, but it will be seen that, in analysing the results, it is not necessary to assume any definite origin for t or ϕ.

† Here arc tan (A tan x) is determined in such a way that it changes continuously as x increases indefinitely.

The curves of fig. 11 were calculated from formula (4.06), assuming α_1 and β_1 constant, and p_1, p_2 proportional to t. They show the type of curve on which the observed values of ϕ may be expected to lie.

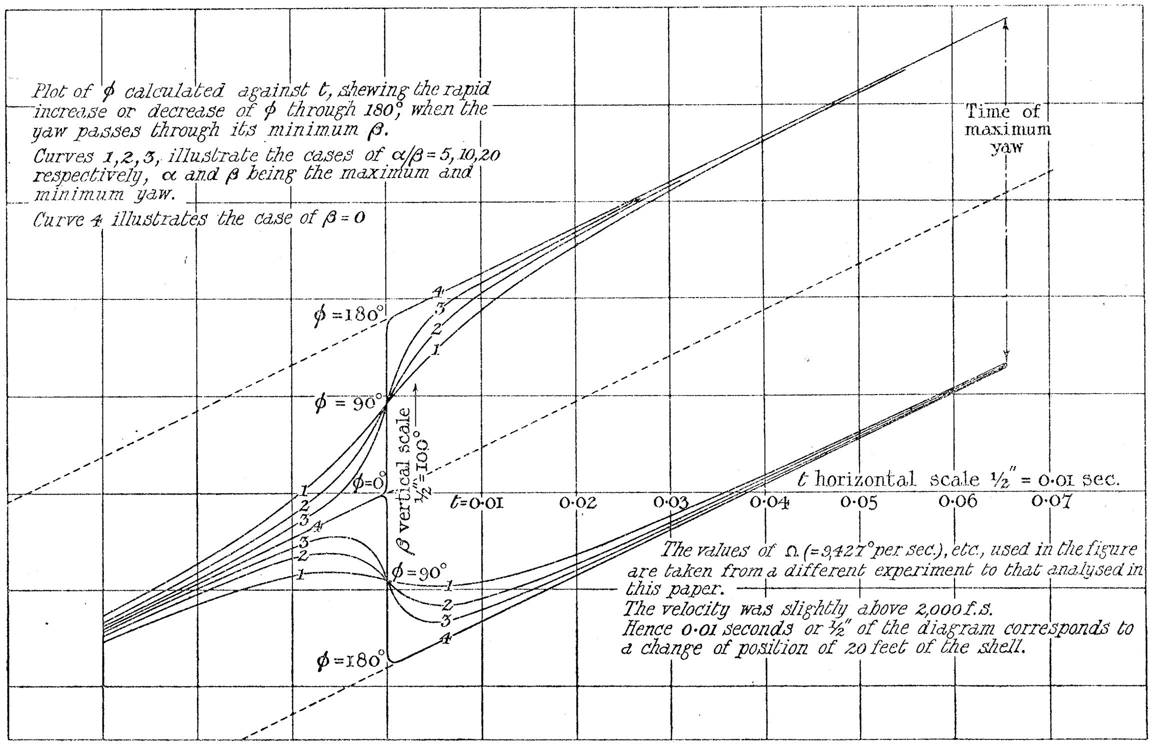


Fig. 11.

§ 4.1. *Analysis of the Experimental Results.*

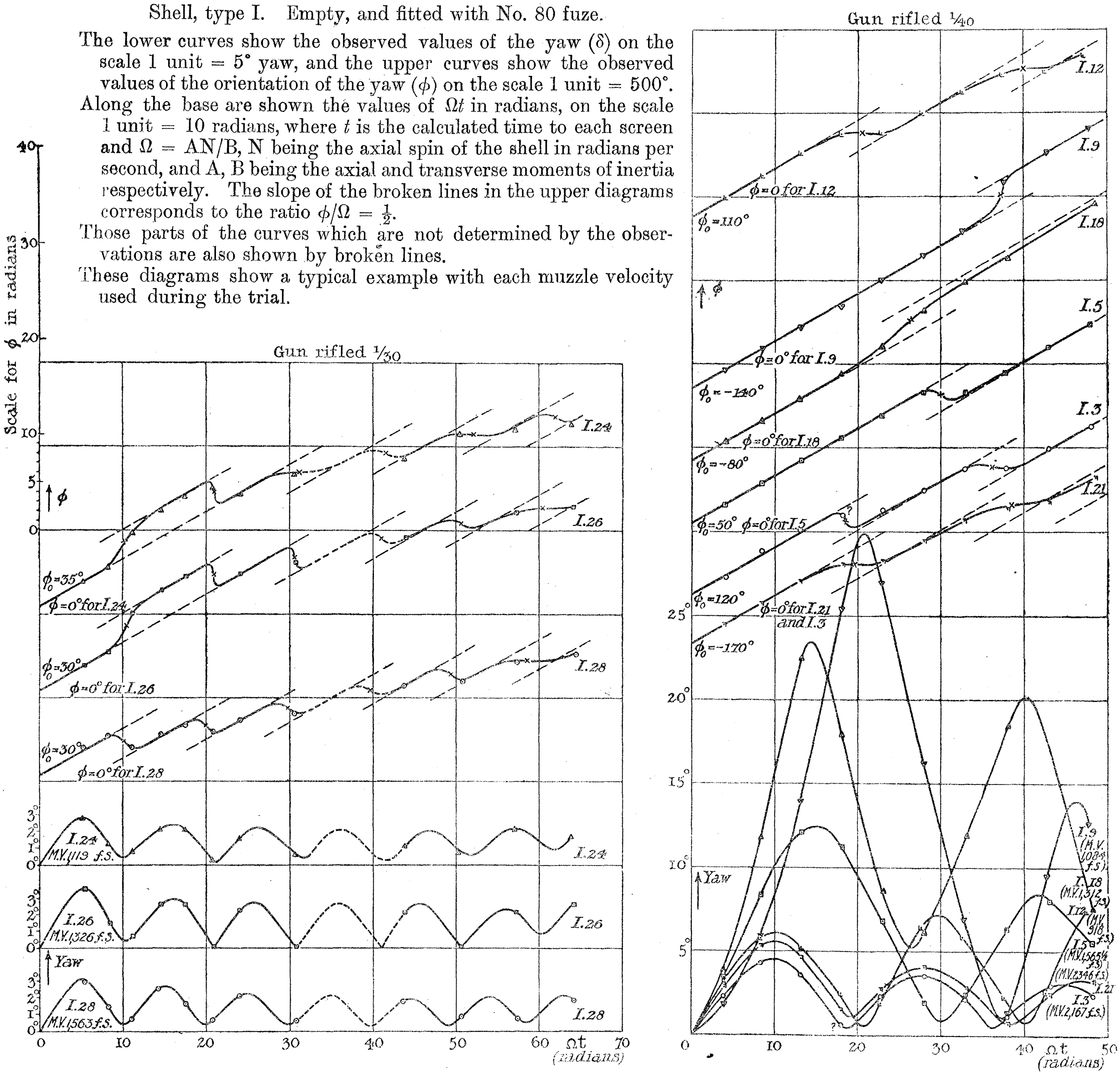
It is now necessary to make use of these results to analyse the experiments. The analysis was carried out by graphical methods. The observed values of δ and ϕ were plotted on separate diagrams, examples of which are shown in fig. 12, against the abscissa Ωt, Ω being determined from the muzzle velocity and the observed moments of inertia. The constant factor Ω was inserted to make the independent variable of zero dimensions; the values of the variable Ωt, at given distances down the range, are also independent of small changes in the muzzle velocity. The observed values of δ are sufficient to give a good determination of curves showing the relation between δ and Ωt, except in the neighbourhood of the minima β, where rapid changes of curvature occur when β is small.* These curves give approximate values of the periods from minimum to minimum, and also the best determination available of the values and times of occurrence T_n of the maxima α. By drawing smooth (non-periodic) curves through the values of α we determine α_1 as a function of Ωt.

* When β is small it may be convenient, in a preliminary plot, to change the sign of δ in each alternate period, so as to obtain smooth curves with δ passing through zero periodically.

In drawing curves for ϕ against Ωt it is necessary to resolve the ambiguities of amount $2n\pi$ as follows:—Equation (4.06), or fig. 11, shows that ϕ increases by

Fig. 12A. Jump card trial, January and February, 1919.
3-in. 20-cwt. gun with 16-lb. H.E. shell, Mark IIB.
Shell, type I. Empty, and fitted with No. 80 fuze.

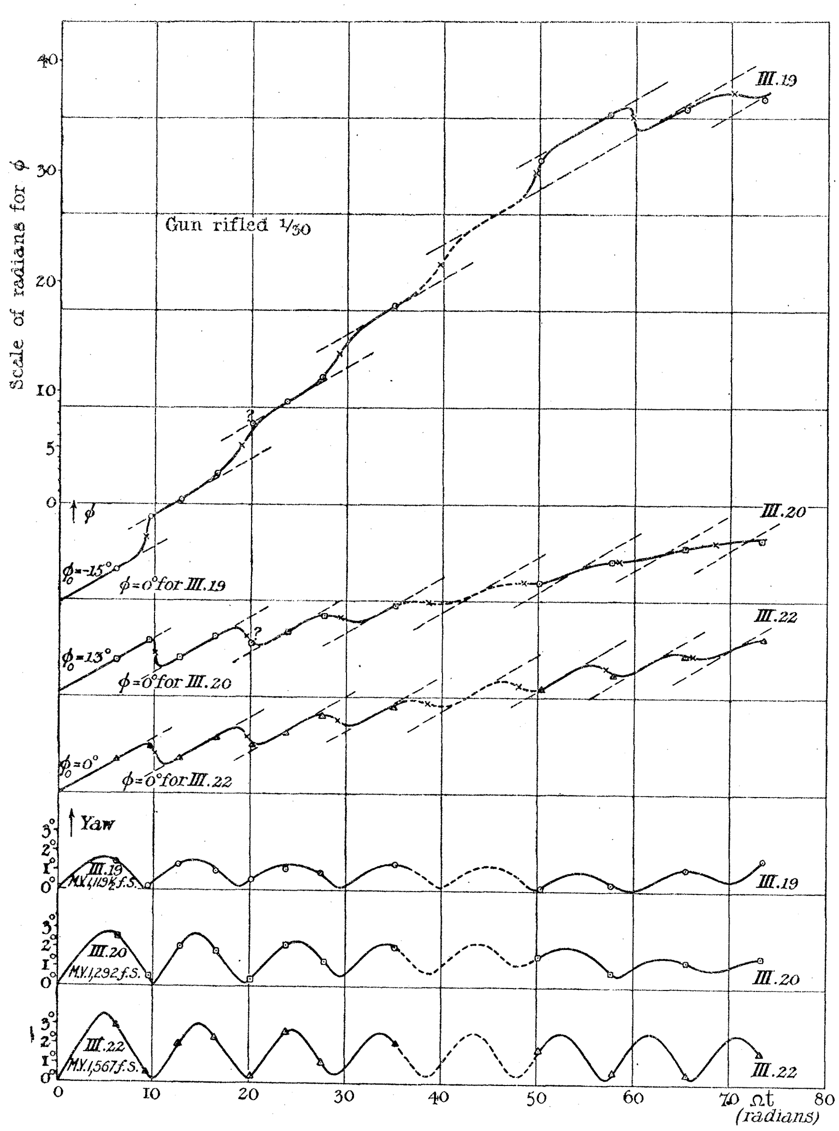
The lower curves show the observed values of the yaw (δ) on the scale 1 unit = 5° yaw, and the upper curves show the observed values of the orientation of the yaw (ϕ) on the scale 1 unit = 500°.
Along the base are shown the values of Ωt in radians, on the scale 1 unit = 10 radians, where t is the calculated time to each screen and Ω = AN/B, N being the axial spin of the shell in radians per second, and A, B being the axial and transverse moments of inertia respectively. The slope of the broken lines in the upper diagrams corresponds to the ratio $\phi/\Omega = \frac{1}{2}$.
Those parts of the curves which are not determined by the observations are also shown by broken lines.
These diagrams show a typical example with each muzzle velocity used during the trial.

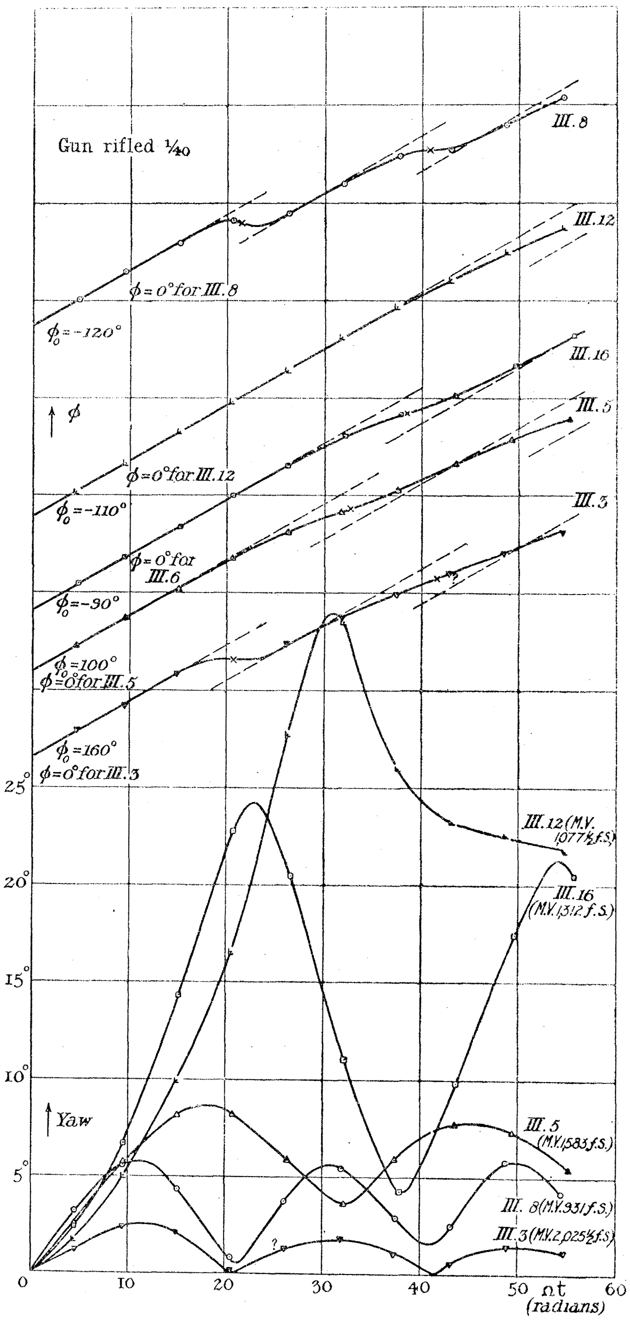


$\frac{1}{2}\Omega T \pm \pi$ in each period T, the increase of $\pm\pi$ occurring chiefly in the neighbourhood of the minimum, especially when β_1/α_1 is small. Hence, by adding to the observed ϕ

multiples of π, which are alternatively odd and even in successive periods of δ, the points can be fitted roughly to a straight line of constant slope. All the points will lie fairly well on this straight line, except those in the immediate neighbourhood of

Fig. 12B. Jump card trial, January and February, 1919. Shell, type III, weighted to make the centre of gravity as far back as possible. Fitted with No. 80 fuze.





the minima of δ. By producing this straight line backwards we can determine the initial value of ϕ. The slope of the straight line determines an independent value of Ω, which is equal to the value deduced from the muzzle velocity if the slope is $0 \cdot 5$.

In practice the values of Ω, obtained by the two methods, were in satisfactory agreement, except for the shells with centre of gravity forward, whose dynamical constants were considerably altered by the set back of the lead block on firing. For these shells the slope of the observed ϕ-curve was taken as defining Ω. The value of B after firing was deduced from this value, and the position of the centre of gravity was determined by equations (2.21) and (2.22).

The curve showing the true relation of ϕ to Ωt must pass through the true values of ϕ, which differ from the observed values only by integral multiples of 2π. It remains doubtful whether the value of ϕ increases or decreases by the amount π radians in passing through a minimum of δ, in addition to its steady increase at rate $\frac{1}{2}\Omega$. This question is settled by the divergence of points near the minimum of δ from the straight line of fig. 11; thus, if the points lie above the straight line in approaching a minimum, there will be an increase of amount π, and *vice versâ*. In this way a continuous curve may be drawn which is consistent with the equation (4.06). Specimens of the curves obtained in the analysis of the actual observations are shown in fig. 12. The portions of the curves in the neighbourhood of the maxima will then coincide approximately with a series of parallel straight lines at distances apart of π radians. The method can only fail in one case when none of the points diverge appreciably from the straight line of fig. 11. This indicates that the value of the minimum β is indistinguishable from zero, while the value of ϕ changes almost discontinuously by $\pm\pi$ at the time of the minimum. It is then immaterial whether the change is taken to be positive or negative.*

The observed values of ϕ in the neighbourhood of the minimum also yield information as to the value of β_1/α_1 and the instant at which the minimum occurs. Let P be any observed point on a ϕ-curve which diverges measurably from the nearest straight portion of the ϕ-curve; lying above it by Δ degrees. Let t_0 be the time of occurrence of the nearest minimum, and δp_2 the change in p_2 between the minimum and P. Then, by (4.06),

$$\cot \Delta = \frac{\alpha_1(t_0)}{\beta_1(t_0)} \cot \delta p_2. \tag{4.101}$$

If, in this equation, Δ, t_0, δp_2, and $\alpha_1(t_0)$ are regarded as known, we can at once obtain a value of β. By adjusting the value of t_0 we attempt to reconcile the one or more values of β obtained in this manner and also the value demanded by the δ-curve. By combining all the available evidence in this manner, remembering that the δ-curve is nearly symmetrical about a minimum, and the ϕ-curve at the same time halfway between two straight portions, we can draw fairly precise final curves,

* The rapid changes or discontinuities in the values of ϕ and δ', which occur when δ is very small or zero, are due to the singularity which occurs at the origin of polar co-ordinates. The motion of the shell is, of course, in all cases continuous.

obtaining values of $\beta_1(t)$ and the times of occurrence of the minima with some accuracy. Such curves are shown in fig. 12.

The following quantities have now been determined from the observations, viz.: the (assumed constant) value of Ω, the times T'_1, T'_2, &c., of the occurrence of the minima of δ (those are more accurately determined than the times of the maxima), and the values of $\alpha_1(t)$ and $\beta_1(t)$ over the range of the experiments. These values are given in Table V.

4.11. *Derivation of the Various Force Components.*—It remains to derive the values of the various force components. By equations (4.054), (4.011)

$$(4.111) \qquad p_2(T'_n) - p_2(T'_{n-1}) = \pi, \qquad \int_{T'_{n-1}}^{T'_n} \Omega\sigma\, dt = 2\pi,$$

giving, as a sufficient approximation,

$$(4.112) \qquad \Omega\sigma = 2\pi/T \qquad (T = T'_n - T'_{n-1}),$$

$$(4.113) \qquad s = \frac{1}{1-(2\pi/\Omega T)^2},$$

$$(4.114) \qquad f_M\left(\frac{v}{a}\right) = \frac{A^2N^2}{4Bs\rho v^2 r^3},$$

where σ, s, and v correspond to the time $\frac{1}{2}(T'_n + T'_{n-1})$. T is therefore the time between successive minima of δ. The values of s and f_M obtained in this manner, or, in a similar way, taking an average over several periods, with the corresponding values of μ and v/a, are given in Table VI.,* and provide the data on which figs. 4 and 5 and Table I. were constructed.

By comparing the values of f_M for shells of form A, with three different positions of the centre of gravity, the values of f_L were deduced by the formulæ of § 1.13. This deduction was done graphically as shown in fig. 13. According to § 1.13 the relation between f_M and l, the distance of the centre of gravity from the base of the shell, should be linear. Fig. 13 shows that all the observed points lie on straight lines within the limits of error of the observations. The slope of each line determines the value of f_L The values of f_L are shown plotted against v/a in fig. 4.

* For the rounds fired from the gun rifled 1 in 30 the time of the first minimum near the muzzle is, in general, badly determined, and the first period is therefore omitted in determining a mean value for s. For the rounds fired from the gun rifled 1 in 40 the time of the first minimum can be determined with fair accuracy by extrapolation.

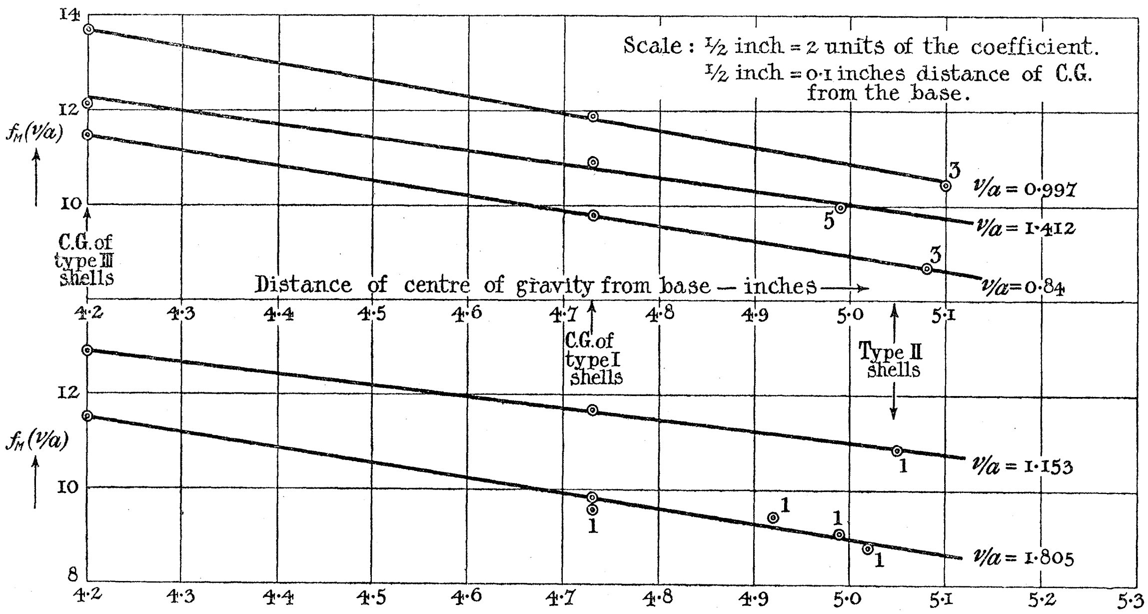


Fig. 13. The determination of the coefficient of the force acting normal to the shell.

The plotted points show the observed values of the couple coefficient plotted against the distance of the centre of gravity from the base.

The slopes of the lines drawn determine the coefficient of the normal force.

The numbers against the points for the Type II. shells give the number of observations whose mean is represented by the plotted point.

4.12. *The Damping Factors.*—It now only remains to derive as much information as possible as to the damping factors κ, h, and γ from the observed values of α_1 and β_1. The factor κ is known in terms of the value of f_L, since, by § 3.1,

$$\kappa = \rho v r^2 f_L / m. \tag{4.121}$$

Squaring and subtracting equations (4.041) and (4.042), we obtain

$$\sigma_0 J^2 e^{-2q_1} = \sigma(\alpha_1^2 - \beta_1^2),$$

$$q_1 = -\tfrac{1}{2}\log\{\sigma(\alpha_1^2-\beta_1^2)\} + \text{const.},$$

$$h+\kappa = \frac{-1}{(t_2-t_1)}\Big[\log\{\sigma(\alpha_1^2-\beta_1^2)\}\Big]_{t_1}^{t_2}. \tag{4.122}$$

In this formula, as well as in those which follow, κ, h, and γ may be treated as sensibly constant over the whole range of one experiment. On dividing (4.041) by (4.042), we obtain

$$\tanh(j-q_2) = \beta_1/\alpha_1. \tag{4.123}$$

Since j and q_2 are both small over the range of the experiments, the formula becomes

$$h-\kappa+2\gamma = \frac{-1}{(t_2-t_1)}\left[\frac{\beta_1}{\alpha_1}\right]_{t_1}^{t_2}. \tag{4.124}$$

The three equations (4.121), (4.122), (4.124) for κ, h, and 2γ are in theory sufficient to determine their values completely. It may be noted again that 2γ is probably negligible and $h-\kappa+2\gamma$ is always positive, so that q_2 continually increases with the time, *and* β_1/α_1 *continually decreases.* The constant j is always very small, but may be positive or negative. If it is positive, β_1 is initially positive, giving the larger average rate of increase of ϕ, which changes to the smaller rate of increase when β_1 becomes negative. If j is negative, ϕ increases at the slower rate from the beginning. Exactly the opposite results would be obtained if $h-\kappa+2\gamma$ were negative. The values of $h+\kappa$ and $h-\kappa+2\gamma$, obtained in this manner, are given in Table VII.

In order to illustrate the actual path traced out by the axis of the shell, it is necessary to plot δ and ϕ as polar co-ordinates. This is done for three rounds in fig. 14. The resulting curves are roughly equivalent to the path of a point on the axis of the shell relative to the centre of gravity. They illustrate the decrease of α_1, the algebraic decrease of β_1, and the tendency to change from quick to slow precession and to settle down to a *steady* slow precession.

The process described above was evolved gradually during the work of analysing the results, so that a number of observations were analysed before it was fully developed. It is probable that if the calculations were to be repeated *ab initio* a number of periods and minima of δ would be slightly altered, but it is unlikely that any serious systematic errors remain.

4.13. *Details of Tables V. to VII.*—The information contained in the General Table of Results, Table V., has been compiled by analysis of the original standard diagrams. As first constructed these were drawn with the time t as abscissa and not Ωt as in fig. 12. It contains practically all the information of importance provided by the more stable shells. In the unstable cases, a number of which occurred during the trial (see for example fig. 12), a detailed study of the whole yaw curve is required which will not be undertaken in this paper.

Column 5 gives the values of the periods of the yaw curve in units of $\frac{1}{1000}$ second. The periods are read off from positions of the minima and sometimes of the maxima. They are entered to the nearest $\frac{1}{2000}$ second. They are in doubt by more than this quantity in many cases, but mainly in the case of the longer periods, in which small errors are of less importance.

Column 6 gives the values of the maxima of the yaw in degrees and decimals to one place of decimals. These values are read straight from the curves and represent roughly the accuracy to which the maxima are in most cases determined by the observations.

Column 7 gives two entries. The first is the value of ΩT for each round, where T is the mean value of the observed period and Ω corresponds to the *observed* value of the steady rate of increase of ϕ from column 4.

The second entry in column 7 is the velocity of the shell at the middle point of the range of periods whose mean value T is used to determine ΩT. The stability factor determined by ΩT is taken to correspond to this velocity. Finally, in column 8, the values of $\beta_1(t)$ are given with their proper sign as determined incidentally in the determination of their times of occurrence (§ 4.11).

The effect of the cards on the observed value of the period and on s, is ignored in Tables V. and VI. The results obtained here are corrected for this effect, as far as possible, before use in Table I. The information given in Table VI. is deduced directly from Table V. by the equations of § 4.11. In certain cases where the yaw was large it was checked by use of the chart of § 3.7.

The total percentage spread of the values of s (or μ) in the group is in most cases satisfactorily small. The value of 6·7 per cent. for the high velocity group of type I. shells is probably partly due to the fact that the fuzes of shells 1 to 4 were slightly damaged before firing in forcing the shells into the cartridge cases.

At a velocity of 1580 f.s. results were obtained with guns of both twists of rifling. The couple deduced from the results for the gun rifled one turn in 40 calibres is, in the cases of shells of types I. and III., slightly smaller than that deduced from the other gun. This is to be expected as the stability in this case is nearly critical and the maxima are rather large (one maximum is as much as 13 degrees for a type I. shell). The solution of § 3.6 can hardly be expected to apply. The next term in the expression for μ of the form $\mu_1 \sin^3 \delta$ may be expected to be becoming appreciable here; apparently its sign is such that it will tend to diminish the observed value of μ, in agreement with wind channel observations (fig. 2). For the shells of type II. the maxima of the yaw are small in both guns and the results are in agreement.

No perceptible dependence of s on the maximum yaw among the rounds of any one group has been detected in these tables.

The agreement between the results for the two guns at this velocity, and between rounds with different maxima of the yaw, is therefore a satisfactory confirmation of the theory.

The values of $h+\kappa$ and $h-\kappa+2\gamma$, deduced from the observations as explained in § 4.12, are given in Table VII. Of these, the former is more reliable as it does not depend on $\beta_1(t)$ which is difficult to determine. The actual values vary considerably from round to round, and only mean values for each group are shown. The results are therefore very rough, but they indicate qualitatively the nature of the damping, which may also be studied in figs. 12 and 14. For example, in fig. 14c, the motion starts with $\beta_1(t)$ positive, so that the loop encloses the origin, O, or point of zero yaw. But since $h-\kappa+2\gamma>0$, $\beta_1(t)$ diminishes and has become negative by the second minimum, the loop failing to reach O. As $\beta_1(t)$ diminishes further, the loop shrinks to a cusp at the fourth minimum, and the motion soon becomes indistinguishable from a precession at the slower rate. In the meantime, the maximum yaw $\alpha_1(t)$ decreases steadily.

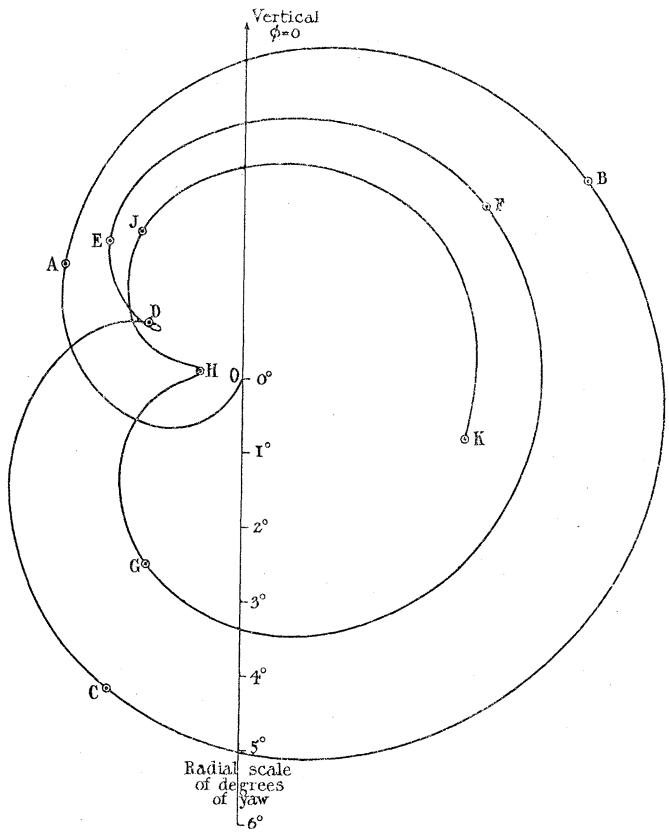


Fig. 14A. Path of nose of shell. Round I.21.

Path described, relative to the centre of gravity, by a point on the axis of the shell in front of the centre of gravity, shown on an enlarged scale.

The total time taken from O to K is 0·2572 second. On the scale used, 1 cm. distance from O represents 1° yaw (very nearly), and corresponds to a linear displacement of 0·118 inch for the nose of the shell from the line of motion of the centre of gravity.

The numerical results for the damping must be affected to some degree by the impacts on the cards, but the available data are not good enough for corrections to be worth making. There is, moreover, the curious phenomenon of an increasing maximum yaw shown by the rounds at 900 f.s. to be accounted for.

The value of κ is known from equation (4.121) and the values of f_L in Table I., so that the damping results determine h and $h+2\gamma$ or, more accurately, $h+2\gamma-\Gamma$ (§ 3.62). It at once appears that $2\gamma-\Gamma$ is negative and of much the same order as h. This is somewhat unexpected. Of course Γ (or $-N'/N$) is positive, but it is hardly likely that its numerical value is much larger than 0·03. It is natural to expect

γ to be small and positive,* which does not fit in with the observations. Further experiments would be needed to throw light on all these points.

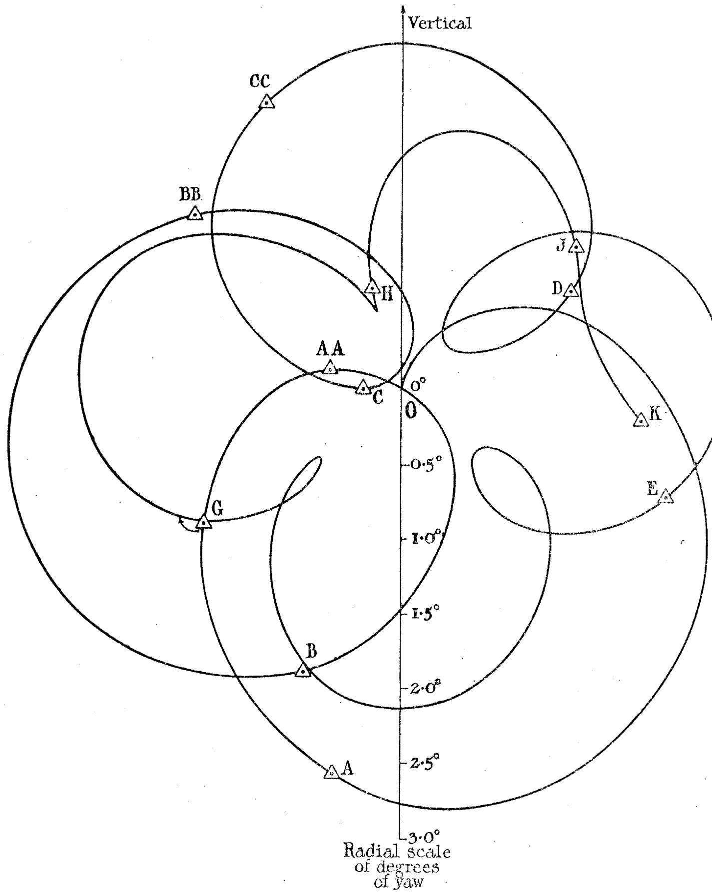


Fig. 14B. Path of nose of shell. Round III.20.

Path described, relative to the centre of gravity, by a point on the axis of the shell in front of the centre of gravity, shown on an enlarged scale.

The total time taken from O to K is 0·3647 second. On the scale used, 2 cm. distance from O represents 1° yaw (very nearly), and corresponds to a linear displacement of 0·128 inch for the nose of the shell from the line of motion of the centre of gravity.

In the fourth column what appears to be the most probable value of h is given; the values of f_H in Table II. are based on these figures and obtained by the equation (see §§ 3.5 and 1.12)

$$f_H = \frac{hB}{\rho v r^4}.$$

* The coefficient γ comes from the swerving couple J (§ 1.12). This couple will only arise if the swerving force K does not act through the centre of gravity. Since the air pressures are greater near the nose than near the base, we may expect K to act *in front* of the centre of gravity. By analogy with the connection between the direction of rotation and the direction of the resulting swerve on a golf or tennis ball at low velocities, we may expect K to act along the axis of M reversed in fig. 9, for a right-handed twist of rifling. This would result in a positive value for γ.

The figures in Table VII. were obtained from the sufficiently stable rounds fired from either gun. In the one comparative pair of groups available the results for the two different stability factors and values of Ω were in agreement.

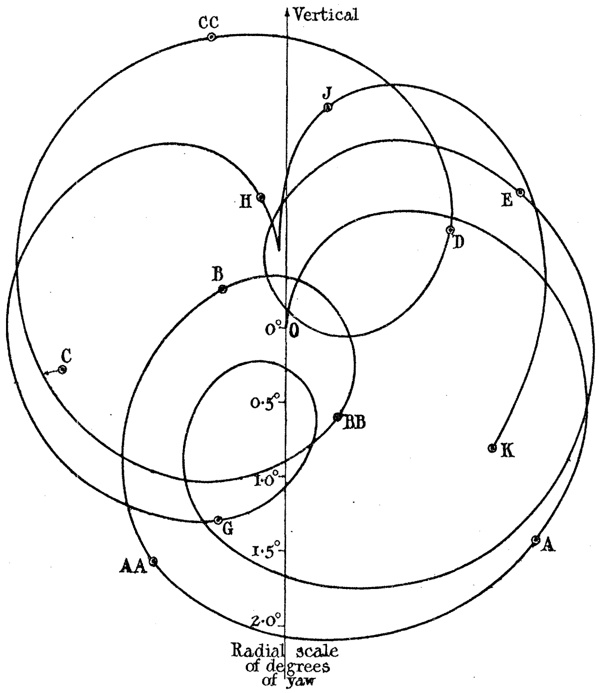


Fig. 14c. Path of nose of shell. Round IV.15.

Path described, relative to the centre of gravity, by a point on the axis of the shell in front of the centre of gravity, shown on an enlarged scale.

The total time taken from O to K is 0·5502 second. On the scale used, 2 cm. distance from O represents 1° yaw (very nearly), and corresponds to a linear displacement of 0·143 inch for the nose of the shell from the line of motion of the centre of gravity.

Note that the first loop encloses O, corresponding to the "stepped up" motion in ϕ. Subsequent loops do not, as the motion in ϕ has changed to the "stepped down" motion. (See fig. 12.)

§ 4.2. *Determination of the Motion of the Shell in Space.*

We now proceed to make use of the results of the experiments to determine the true path of the centre of gravity of a shell projected in a given manner. The solution of the equations of type α is sufficient for this purpose so long as the yaw does not exceed 0·1 radian; the values of f_L, f_M, f_H, &c., which we have obtained, are sufficient to determine the motion completely in this case. Assuming that the maximum yaw due to the initial disturbances is less than 0·1 radian in the first period, it will remain so throughout the trajectory; the yaw arising from the particular integral will not exceed 0·1 radian until the velocity has fallen considerably below 700 feet per second. Hence, when the yaw exceeds 0·1 radian, the wind channel values for the various force components as shown in fig. 2 can be used; it will, however, be necessary to abandon the above method of solution and proceed by the step-by-step integration of the equations of type β.

Throughout the following work all numerical results will be based on a set of plane trajectories of a 16-lb. shell, of external form A, fired at a muzzle velocity of 2000 f.s., calculated by the ordinary ballistic methods.* The various elements of the trajectories at elevations of 30 degrees and 50 degrees and a list of constants for the service shell, to which the calculations apply, are given in Table VIIIA.

From the value of ξ for the general solution, as given in § 3.65, we can deduce the true path of the centre of gravity in terms of the tabulated elements of the plane trajectory. Let $(X_1, Y_1, 0)$ be the co-ordinates of the shell in the plane trajectory at time t, and (X, Y, Z) the corresponding co-ordinates in the true (twisted) trajectory. The direction cosines of the tangents to the two trajectories are $(X'_1/v_1, Y'_1/v_1, 0)$, or $(\cos\theta_1, \sin\theta_1, 0)$ and X'/v, Y'/v, Z'/v, so that, to the usual order of approximation,

$$(4.201)\qquad c\xi = (Y'-Y'_1)(\cos\theta_1)/v_1 - (X'-X'_1)(\sin\theta_1)/v_1 + iZ'/v_1$$
$$= (H'+iZ')/v_1,$$

say, while the condition $v = v_1$ gives

$$(4.202)\qquad (X'-X'_1)\cos\theta_1 + (Y'-Y'_1)\sin\theta_1 = 0.$$

It is convenient to separate the parts of the solution arising from the complementary function and the particular integral. To determine the latter, we use equations (3.632), (3.633), and (4.201), obtaining

$$(4.203)\qquad \frac{Z'}{v_1} = c\int_0^t \frac{-4s\kappa\theta_1'\,dt}{c\Omega}$$
$$= c\psi,$$

say, neglecting the terms $i\theta''_1/\Omega$ in Φ (see § 3.20). This equation defines ψ. Therefore

$$(4.204)\qquad Z = \int_0^t \psi v_1 \cos\theta_1\,dt,$$

where ψ may be written (since $-\theta'_1/c = g/v_1$)

$$\psi = \int_0^t \frac{g\kappa AN}{\mu v_1}\,dt = \frac{Ag}{mr}\int_0^t \frac{N f_L(v/a)}{f_M(v/a)}\frac{dt}{v^2}.$$

To the same approximation $(X'-X'_1)/v_1$ and $(Y'-Y'_1)/v_1$ are $O(1/\Omega^2)$, so that (X_1-X) and (Y_1-Y) are small compared to Z, so long as the approximations hold. The above result is identical in form with the "classical" formula of MAYEVSKI,

* Trajectories were calculated with the ballistic coefficient 1·75.

freed from the unnecessary restriction that f_L/f_M and N should be constants.* We have thus justified the use of the plane trajectory as an approximation to the true motion. The leading terms in $(X-X_1)$ and $(Y-Y_1)$ can be calculated if required.

The effect on the motion of a change in initial conditions is obtained from the complementary function. Equation (3.655) gives the value of ζ corresponding to the general initial conditions $\zeta_0 = 0$, $\eta_0 = a$, $\eta'_0 = b\Omega$, where a and b may be complex. Substituting in equation (4.201) the part of ζ arising from the complementary function, it appears that $H+iZ$ is made up of three parts :—

(*a*) A periodic term

$$H_1+iZ_1 = -\frac{4\kappa v_1}{\Omega^2}\left(\frac{K_1\eta_1}{(1+\sigma)^2}+\frac{K_2\eta_2}{(1-\sigma)^2}\right).$$

(*b*) A term

$$H_2+iZ_2 = -\{K_1(\zeta_1)_0+K_2(\zeta_2)_0\}\int_0^t cv_1\zeta_3\,dt,$$

which is the effect of a variation in the direction of projection, as mentioned in § 3.64.

(*c*) A constant term H_3+iZ_3 equal to the initial value of H_1+iZ_1 with its sign changed.

4.21. *Numerical Results as to Motion of Centre of Gravity.*—The only data as to the forces on the shell required for the calculation of the drift are the value of f_L/f_M as a function of v/a. This is derived from the results of the jump card experiments for $v/a > 0{\cdot}7$, and from wind channel experiments for $v/a < 0{\cdot}7$, and is shown plotted in fig. 15.

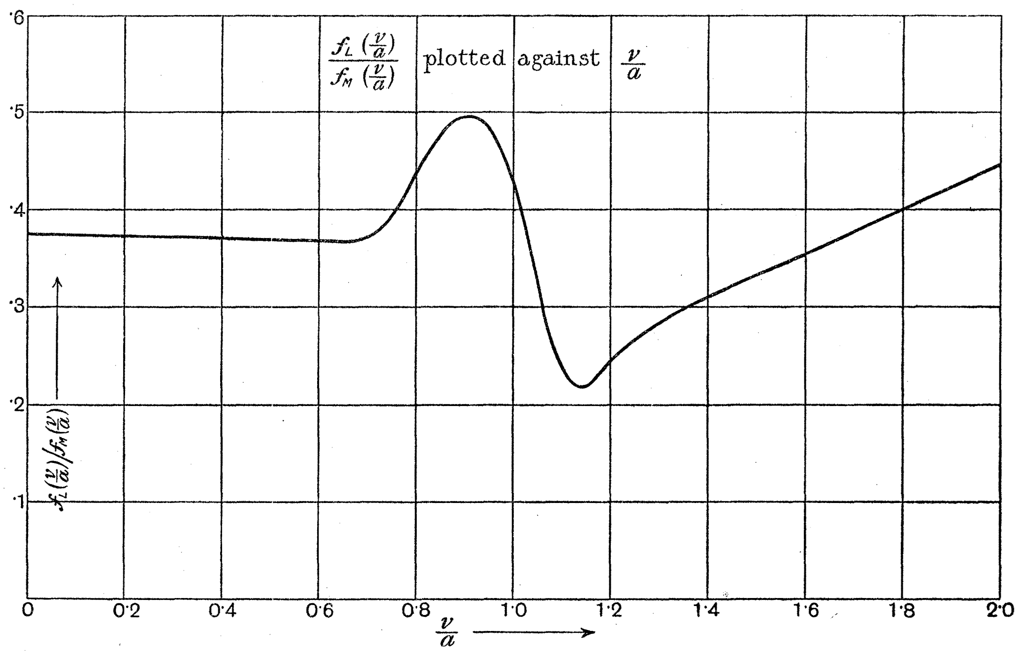


Fig. 15.

* PRESCOTT obtains a solution of the equations of motion in the form of a series of which the first term is also equivalent to MAYEVSKI'S formula. (See Introduction, p. 296.)

As the value of the couple Γ is only known to be small, it is necessary to assume that N is constant. The principal steps in the calculation of the drift Z, by means of (4.203) and (4.204), for the trajectories at 30 degrees and 50 degrees, are given in Table VIIIb. for the gun rifled 1 turn in 30 diameters of the bore. For a different rifling the drift (N constant) is proportional to N.

It is only necessary to estimate roughly the effect of the complementary function on the motion of the shell since the total effect is always fairly small. The periodic term H_1+iZ_1 is obviously smaller than $4\kappa v_1\alpha_1/\Omega^2(1-\sigma)^2$ in absolute value, where α_1 is defined as in § 4.0, equation (4.041). The initial value of the coefficient of α_1 is $1\cdot25$ feet for the gun rifled 1 turn in 30 diameters of the bore; both α_1 and its coefficient diminish rapidly. Taking $\alpha_1 = 0\cdot1$ radian as an extreme case,

$$|H_1+iZ_1| < 1\cdot5 \text{ inches.}$$

The actual value in practice is probably always $< 0\cdot5$ inch, which is small. It explains why no evidence of helical motion was obtained in the jump card experiments. The constant value of $|H_3+iZ_3|$ is equal to the initial value of $|H_1+iZ_1|$ and is also negligible. There remains only the term H_2+iZ_2. This is equivalent to an angular deviation of $|K_1(c\xi_1)_0+K_2(c\xi_2)_0|$, which is less than $2\kappa\alpha_1/\Omega(1-\sigma)$. The coefficient of α_1 for the gun rifled 1 turn in 30 diameters of the bore is $1\cdot8\times10^{-2}$, so that for a value of α_1 of 0·1 radian the angular deviation is of the order $0° \, 6'$. This is of the same order of magnitude as the angular jump likely to be due to changes of form and position in the gun and mounting under firing stresses. When it varies from round to round in magnitude and direction, it will account for an *irregularity* of the corresponding amount in the observed positions of the shells at any time. When, as may sometimes be the case, it remains fairly constant from round to round, it will cause systematic errors in the position of the shell at any time. It is probable that anomalous values of the drift, sometimes observed at short times, are due to this cause. Practical results, however, more often fully justify the use of the particular integral alone to give a mean value of the drift when the initial disturbance is only known to be small.

The results of the above calculations of drift will now be compared with observations of the Z co-ordinates of the bursts of shells, fired at Portsmouth, at corresponding elevations, in February and April, 1918. For this purpose use is made of the azimuth of the shell burst Z/X; the quantity $A = Z/Xt$ is tabulated, since its value varies slowly along the trajectory (Table IX.). The agreement between observation and calculation is as good as could be expected, in view of the uncertainty in the wind effects, and provides important evidence as to the correctness of the whole theory.

4.22. *The Damping of the Angular Oscillations and the Effect on the Head Resistance.*—We have now obtained the complete motion of the centre of gravity of the shell by use of the equations of type α for the two standard trajectories; we have, in so doing, assumed that the velocity of the shell is the same in the plane and true trajectories; we must now examine more closely the possible effect on the drag of the angular oscillations and their rate of damping, by means of the values of h and κ obtained above. From equations (4.031) and (4.032) it appears that for sufficiently large values of t, δ and ϕ are given approximately by the equations

$$\delta = \tfrac{1}{2}J(\sigma_0/\sigma)^{\frac{1}{2}}e^{-(q_1-q_2)},$$
$$\phi = \phi_0 + (p_1 - p_2),$$

so that the shell settles down to a steady precession with the slower precessional angular velocity, the yaw gradually diminishing in proportion to the factor $(\sigma_0/\sigma)^{\frac{1}{2}}e^{-(q_1-q_2)}$. This quantity is tabulated in column 9, Table VIIIB., on the assumption that $\gamma = 0$ and $h = 3\kappa$. The damping is actually more rapid than is indicated by this approximation.

The question of the rate of damping of the initial oscillations of a shell is of importance on account of its effect on the drag R, for the effect, though it may be small, will be cumulative, since it tends always to increase R. If it is assumed that the effect on R is given by*

$$\text{(4.221)} \qquad R = R_0(1 + k\delta^2),$$

where R_0 is a function of v and k is a constant, it is possible to obtain an approximate formula for the total change in velocity produced on the assumption that the time taken to damp out the oscillations is relatively small. We have

$$\text{(4.222)} \qquad -\Delta v = \frac{1}{m}\int_0^\infty (R - R_0)\,dt,$$
$$= \frac{kR_0}{m}\int_0^\infty \delta^2\,dt.$$

Using (4.031), and integrating on the assumption that σ is constant, q_1, q_2, and p proportional to t, and j zero, we obtain

$$\text{(4.223)} \qquad -\Delta v = \frac{kJ^2R_0}{2m}\int_0^\infty e^{-2q_1}(\cosh 2q_2 - \cos 2p_2)\,dt,$$
$$= \frac{kJ^2R_0\sigma_0^2(h_0+\kappa_0)}{2m\{\sigma_0^2(h_0+\kappa_0)^2 - (h_0-\kappa_0+2\gamma_0)^2\}}.$$

At present we have no information as to the value of k except at low velocities, while J varies from round to round so that no numerical results can be given. It seems likely that this is a cause of irregularities in range in practice of first class importance. The yaw arising from the particular integral will also tend to increase the resistance, but the effect is of less importance in a low angle trajectory.

* By symmetry, there can be no odd powers of δ in R.

4.23. *The Exact Motion in a High Angle Trajectory.*—It will be shown in the next section that, for a trajectory of much higher elevation than 50 degrees, the approximations for the particular integral break down, and the equations of type α, are not applicable to the later stages of the trajectory when the velocity has fallen much below 500 f.s. These later stages occur after the initial oscillations have been damped out, and are suitable for the use of equations of type β. These equations can be integrated step-by-step on the basis of the wind channel values of R, L, and M (fig. 2), which apply to velocities up to 700 f.s. The process is analogous to the usual method of calculating a plane trajectory, but rather more laborious, and has been carried out in one case only for a 3-inch $12\frac{1}{2}$-lb. shell fired at 70 degrees with a muzzle velocity of 2450 f.s. At 40 seconds the yaw has reached the large value of 60 degrees and is still increasing. This is partly due to the large initial value of the stability factor (about $4\cdot0$) indicating that the spin is unnecessarily large for this shell. The results of comparing the drift with observation were again fairly satisfactory in this case; but details of these results cannot be given here.

§ 4.3. *Estimate of the Errors in the Various Solutions.*

In the development of the various solutions of the equations of motion in Part III., it was found necessary to neglect certain terms. We shall now proceed to examine these terms in succession, and to determine, as far as possible, their numerical values, using the values of the various force components obtained from our experiments. By so doing we shall justify the use of the solutions by showing that the terms neglected are all very small over the range covered by the jump card experiments. In the applications to the later parts of a trajectory, the solutions break down in certain cases, and an examination of the error terms enables us to define the circumstances under which the solutions are valid. We proceed to examine the various terms. It is necessary as a rule to distinguish the terms neglected in obtaining the complementary function from the terms neglected in obtaining the particular integral.

In the complementary function, m, n, y, z are periodic functions of the time with periods comparable with Ω. For the solution to be applicable we have to assume that δ is always small (say $\delta < 0\cdot1$ radian). Then m, n, y, z are all small quantities comparable with δ, and m'/Ω, m''/Ω^2, &c., are also comparable with δ, while $(1-l)$, l'/Ω, l''/Ω^2, &c., are of the order of δ^2. In neglecting terms independent of θ'_1 from the equations (3.202), (3.203), we are guided by the condition that all terms neglected should be of the order of δ^2 compared with those retained. As regards the terms containing θ'_1 or θ''_1, it appears that the maximum value of θ'_1/Ω in the 50 degrees trajectory (for rifling 1 in 30) is 30×10^{-5}, its initial value being 5×10^{-5}. Hence all terms such as $nm'\theta'_1$, $n\theta'^2_1$ are completely negligible in obtaining the complementary function.

If all terms in θ'_1, θ''_1 are removed from equations (3.202), (3.203), they become equivalent to the equations of type γ; the errors in neglecting further terms may, therefore, be determined by comparing the solutions of equations (3.204), (3.205), &c., of type α with those of equations (3.404), (3.405), of type γ (assuming μ constant in both cases). Equations (3.7041), (3.7043), (3.7044), (3.705) can be made to give the following approximation to the true value of s in terms of T and α:—

$$s = \frac{1}{1-4\pi^2/\Omega^2\mathrm{T}^2}\{1-\tfrac{1}{8}(2s+1)\alpha^2\}.$$

This is valid so long as (1) α is so small that α^4 may be neglected, and (2) $s-1$ is positive and large compared with α^2. Comparing this with the corresponding first approximation (4.113), we obtain the error in the value of s due to the neglect of the terms in $(1-l)$, l'', &c., in equations (3.202), (3.203). The relative value of the error is given in the following table:—

α	$s = 1\cdot1$.	$s = 2$.	$s = 3$.
10°	0·012	0·019	0·027
5°	0·0030	0·0048	0·0066
2·5°	0·0007	0·0012	0·0016

In analysing the jump card trial, whenever the error from this cause is appreciable, the results have been corrected by determining the values of s from the chart described in § 3.7.

It appears also from the solution of the equations of type γ that when $s \leq 1$ the initial angular motion is still periodic, but no longer of the nature of a small oscillation, since the period is a function of the amplitude and tends to infinity as the initial disturbance tends to zero.

In using equations (3.202), (3.203) to obtain the particular integral, the order of magnitude of the various terms is different. The term $\mathrm{AN}l\theta'_1$ is now the most important, while n is $\mathrm{O}(1/\Omega)$ and m is $\mathrm{O}(1/\Omega^2)$ with the notation of § 3.6. Most of the terms neglected are then $\mathrm{O}(1/\Omega^4)$ compared to the principal term, and completely insignificant, but $\mathrm{B}nl\theta'^2_1$ is $\mathrm{O}(1/\Omega^2)$ and would affect the third term in the expansion for $\bar{\eta}$. Its effect however is completely negligible.

4.31. *The Equations of Motion of the Centre of Gravity.*—These equations may be treated in a similar manner. In obtaining the complementary function, y and z are small compared to m and n (see equations (3.624), (3.625)), κ/Ω being initially less than 0·01. As regards the differential equation for u (3.2141), the effect of neglecting the terms arising from the variation of R with δ has been discussed in § 4.22; no numerical data are available; the effect is theoretically second order. The term in $1-x$ is obviously negligible. Omitting these terms, the equation can be reduced to the form

$$u'+\alpha u = -gy\cos\theta_1,$$

where

$$\alpha = \frac{1}{m^*}\frac{\partial}{\partial v_1}\{\mathrm{R}(v_1+\theta u,\ \theta\delta)\} \qquad (0<\theta<1).$$

As a rough approximation we may assume that $\mathrm{R} = kv^2$, so that $\alpha = -2v'_1/v_1$. We then find that

$$\frac{d}{dt}\frac{u}{v_1^2} = -\frac{yg\cos\theta}{v_1^2}.$$

In the case of the complementary function, y consists of a constant term less than 2×10^{-3}, and periodic terms whose period is of order $1/\Omega$.

The former makes a contribution to u/v_1 which is still less than 10^{-3} after 20 seconds. The latter contributes† a term of order $yg/v_1\Omega$ which is always less than 3×10^{-6}.

In the case of the particular integral y is $\mathrm{O}(1/\Omega^2)$, see § 4.2. Hence in all cases we are justified in putting $u = 0$, $v = v_1$, so long as the equations of type α hold at all, with the proviso that this conclusion may be at fault if the k of § 4.22 is numerically large.

In reducing equations (3.212) and (3.213) we put $x = 1$, $\cos\delta = 1$. This amounts to neglecting $1-x$, $1-\cos\delta$ compared to 1, and is obviously justifiable. We omit altogether from (3.212) the terms $x\theta'_1+(g/v)\cos\theta_1$, or $-g\cos\theta_1\ (x/v_1-1/v)$. This term is excessively small, but could be retained, if desired. Finally we omit the terms in $y\cos\theta_1$, justifying the omission by the arguments used above for the same term in the equation for u.

§ 4.32. *Errors in the Solution for the Complementary Function.*—The second term R_1 in the expansion of R in equation (3.6233) will be taken as representing the principal part of the error in the standard solution for the complementary function arising at this stage. Its value is

$$\mathrm{R}_1 = -\tfrac{1}{4}\mathrm{M}^{-\frac{3}{4}}\frac{d^2}{dt^2}(\mathrm{M}^{-\frac{1}{4}}),$$

where M has the value appropriate to (3.6231). For simplicity in estimating errors we may take only the leading term in M so that here

$$\mathrm{M} = \tfrac{1}{4}\Omega^2\sigma^2.$$

The values of s determined from the jump card trial and the data of the 50 degrees plane trajectory are tabulated in column 2 of Table VIIIB. The value of σ can be

† This contribution is of the form $\int_0^t f(t)\,e^{i\Omega t}\,dt$, which is of the order $(1/\Omega)\times$ (maximum of $f(t)$) under suitable restrictions on $f(t)$, which are satisfied here.

deduced and its first and second differential coefficients obtained from a difference formula. In this way we find that initially

$$R_1/R_0 = 0\cdot000011,$$
$$R_1/R_0 = 0\cdot0031,$$

for the guns rifled 1 turn in 30 diameters and 40 diameters respectively. Moreover, the value of R_1/R_0 diminishes along the trajectory. The neglect of this term is therefore justified, provided $s > 1\cdot1$, and the total error in the solution will probably be of the same numerical order.

The contribution of ζ (see 3.6235) to the coefficient of η in equation (3.613) is

$$-\frac{2\,\kappa c'}{c\,(1\pm\sigma)},$$

the principal term in this coefficient being $-\Omega^2/4s$. The relative value of the error in omitting this term is therefore

$$\left|\frac{8sc'\kappa}{c\Omega^2\,(1\pm\sigma)}\right|,$$

which is less than

$$\left|\frac{4s\kappa}{\Omega}\,\frac{4s\theta'_1}{\Omega}\tan\theta_1\right|.$$

The values of these three factors can be obtained from Table VIII. The maximum value of this ratio for the 50 degrees trajectory is $\frac{1}{250}\tan\theta_1$. This is negligible.

In evaluating M to obtain equation (3.6234), terms such as κ^2/Ω^2, $\kappa h/\Omega^2$, κ'/Ω^2 are neglected. It is unnecessary to evaluate such terms in detail, since it is known that κ/Ω and h/Ω are less than $0\cdot02$ in all cases. It would, however, be easy to write down equation (3.6234) with such terms included.

4.33. *Errors in the Particular Integral.*—The errors of the expression for the motion of the centre of gravity of the shell, given in (3.632) and (3.633), may be obtained from the expansion of the particular integral in powers of $1/\Omega$. The ratio $i\theta''_1/\Omega\theta'_1$ of the two terms in Φ, § 3.2, can be worked out from the data of the plane trajectory. Its initial and greatest value for the gun rifled 1 turn in 30 diameters is $(0\cdot0008)$, so that the second term is entirely negligible in comparison with the first. Writing therefore $\Phi_1 = \theta'_1$, it appears that the terms of order $1/\Omega^2$ in (3.632) are real and so do not affect the drift. The next term is (with $\gamma = 0$)

$$\frac{4s}{(i\Omega)^3}\left\{-\eta''_1+\frac{d}{dt}\,(4s\eta'_1)+\eta'_1\,(8s\kappa-h-\kappa)+\eta_1\left[\frac{d}{dt}\,(4s\kappa)+4s\kappa^2-h\kappa-\kappa'\right]\right\},$$

where $\eta_1\ (= 4s\theta'_1)$ is the coefficient of $1/i\Omega$ in the first term in the expansion of $\bar{\eta}$.

There are also a number of other terms involving c', c'' and c'^2. The terms in c' are very small initially and vanish at the vertex, so that they are never likely to become important. The other terms in c'' are certainly very small provided that s is of order unity. Since s varies roughly inversely as the square of the velocity (*i.e.*, f_M constant), the magnitude of the terms containing s rises very rapidly in the later stages of the trajectory when v becomes small. The first term in $\bar{\eta}$, $-4is\theta'_1/\Omega$, is given numerically in Table VIII., where it appears how rapidly it increases as the velocity falls. The values of the second term as given in equation (3.632) are also given (Table VIIIB., column 8). It appears that the ratio of the second term to the first term is always small so long as the first term is small. This term represents the effect of the particular integral in altering the co-ordinates in the plane of fire. The third term as given above is more difficult to evaluate, and only a rough estimate has been made of its value at two points on the 50 degrees trajectory. The results are:—

Seconds.	Third term/first term.	Third term.
$t = 0$	$-2\cdot 02 \times 10^{-3}$	$-8\cdot 5 \times 10^{-7}$
$t = 20$	$-1\cdot 94 \times 10^{-2}$	$-7\cdot 2 \times 10^{-4}$

The value of the drift as estimated by the first term is therefore slightly too large. The first part of the third term, $-4s\eta''_1/(i\Omega)^3$, is of special interest, as it represents the sole contribution of the term η'' in equation (3.613) to the value of $\bar{\eta}$ to this order. The term η'' represents all that remains in the equations of type α of the terms in B neglected in § 3.3 in obtaining the equations of type β. The initial value of $-4s\eta''_1/(i\Omega)^3$ is only $3{\cdot}46 \times 10^{-5}$ of the first term in $\bar{\eta}$ in the 50 degrees trajectory, and this ratio does not tend to increase as the velocity diminishes. This makes it likely that the equations of type β give an accurate solution in all cases when the initial conditions are those of the particular integral.

Returning to the particular integral, we have shown that the third term is only $0\cdot 03$ (?) of the first term at the vertex of a 50 degrees trajectory where the velocity is as low as 500 f.s. For a trajectory at still higher elevation the minimum velocity is lower; the value of the first term soon becomes too great for the use of approximations which neglect $1-\cos\delta$, while the third term can no longer be neglected in comparison with the first term. The solution therefore fails when the elevation much exceeds 50 degrees as soon as the velocity has fallen much below 500 f.s. The true degree of approximation given by the expansion can only be obtained in a special case. If the terms in η'' in equation (3.613), and terms of the solution containing c', &c., arising from the terms in ζ are neglected, it may be shown that the error of the expansion at any stage is less in numerical value than the last term retained.* Hence the numerical estimates of the third term, obtained above, justify the use of the first term only to obtain an approximate value of the drift at all elevations up to 50 degrees and for the initial part of a trajectory at any elevation.

PART V.—SUMMARY AND CONCLUSION.

§ 5.0. *Summary of Preceding Results.*

In the earlier parts of this paper we have suggested a tentative set of components for the complete force system acting on a shell moving through air (or other medium), in which this complete system may be assumed to depend at any moment only on the position and velocities of the shell. We have submitted these suggestions to the test of experiment, and found that, so far as we have carried the analysis in this paper, the experiments confirm our suggestions, and provide, when the yaw is small, numerical values for two of the force coefficients (f_M with a probable error of 2 per cent. and f_L with a probable error of 10 per cent.) for velocities up to double the velocity of sound. Rough values for a third coefficient f_H are also provided. It appears probable that the other components (except of course the drag) are much less important, and that values of the yaw up to perhaps 10 degrees may be regarded as small in this connection.

It is convenient to summarize here what we do and do not know about the components of the force system on the shells used in this trial. The values of the drag coefficient f_R may be regarded as known for all velocities at zero yaw. The values of f_M and f_L are roughly known for velocities up to $v/a = 2 \cdot 0$, and values of the yaw less than 10 degrees. From wind channel experiments f_R, f_M and f_L are all known for all values of the yaw when v/a is small, and these determinations probably apply so long as $v/a < 0 \cdot 7$. The damping effects are only known roughly, but sufficient is known to estimate how long a shell will take effectively to settle down to a steady state of motion.

On the other hand the variation of f_R with yaw is entirely unknown except from wind channel experiments, and so is the variation of f_M and f_L at values of the yaw greater than 10 degrees. The rate of diminution of the axial spin is unknown and so is the size of the swerve effect, though this latter is not likely to be important.

The variation of f_R with yaw could be studied experimentally by a suitable combination of jump card observations, with the use of the solenoid chronograph to determine as exactly as possible the deceleration of the shell at every point. The values of f_M and f_L for larger values of the yaw could be obtained by a detailed analysis of unstable rounds in which large values of the yaw are realized. A start

* The equation is now of the first order in η only, so that the exact solution may be written down in the form of an integral. By successive integration by parts we obtain the expansion (3.632) together with an integral representing the error after n terms.

could be made with the data of the present trial, but we cannot undertake this in this paper.

In Part III., we have arrived at two separate solutions of the equations of motion of a shell treated as a rigid body, which together cover practically all types of motion which are likely to occur in practical shooting. (We ignore here the case of an unstable shell, since it is of no practical use.) A general solution of the equations of motion of type α has been developed, which applies with sufficient accuracy to the most general type of motion of a shell whose angle of yaw δ and inclination of the tangents of true and plane trajectories do not exceed (say) $0 \cdot 1$ radian. The solution of the equations of type β can be applied with sufficient accuracy to the steady (non-oscillatory) motion of a shell at any angle of yaw. In practice the large angles of yaw ($> 0 \cdot 1$ radian) only occur in the neighbourhood of or beyond the vertex of a high angle trajectory, and by this stage the initial angular oscillations of the shell have been completely damped out so that the condition for the applicability of the solution of type β is satisfied. Thus the solutions we have obtained, though theoretically inadequate, are probably sufficient to cover all cases likely to occur in practice.

In order to make use of these solutions to determine the complete motion of a shell, information is necessary as to the complete force system acting on the shell. Our information, as we have seen, is fairly complete for angles of yaw up to 10 degrees, and can be applied to calculate the true trajectory of any shell for which the angle of yaw does not exceed this value, if the loss of spin and increase of drag with yaw can be ignored.

Larger angles of yaw (exceeding 10 degrees) occur in general only as a consequence of the low velocity of the shell near the vertex of a high angle trajectory. The force system is then mainly covered by wind channel observations. The information as to the force system obtained by our methods is thus adequate for the calculation of a complete set of twisted trajectories at all elevations, at any rate for a 3-inch shell.

§ 5.1. *Problems for Further Discussion.*

5.11. *Unstable Rounds.*—We have already mentioned that further information about f_M and f_L, at yaws greater than 10 degrees, could be obtained by analysis of the unstable rounds. This requires the application of the exact top equation with a variable value of μ (§ 3.7) to the discussion. No means of introducing damping effects into these equations has yet been devised. It should, however, be possible to obtain fairly reliable information as to the variation of f_M and f_L with yaw between the angles of 10 degrees and 30 degrees by the analysis of the unstable rounds fired in this trial (Table IV.).

5.12. *Initial Conditions.*—By extrapolating the δ-curve and ϕ-curve backwards to the gun muzzle ($t = 0$) information may be obtained as to the way in which the projectile leaves the gun, which may prove of value. Owing to the effect of the initial oscillations on the ranging of the shell, it is important to determine whether, in general, the initial disturbance takes place at, or nearly at, the same orientation. Secondly, it is important to determine whether the initial oscillations may be regarded as caused by an impulsive couple whose size is independent of the twist of the rifling. If this is so, the amplitude of the initial oscillations of a shell can be cut down indefinitely by sufficiently increasing the spin. If, however, as appears to be the case from a rough survey of the data of the present trial, the initial circumstances are such that the impulsive couple (or its equivalent) increases in proportion to the twist of the rifling, then no increase of spin will reduce the oscillation below a certain definite limit. This conclusion would be technically important, as in the later stages of flight the spin is always in excess of requirements, and so the initial spin should be kept down as much as possible.

5.13. *Wind Effects.*—In calculating the effect of wind on a shell it is usual to assume that the shell at once turns its nose to the relative wind. This is not strictly correct, and the true angular motion in a wind when the velocity is known at every point can be determined by our theory, since the forces acting on the shell depend only on its motion *relative* to the air. Consider, for example, the special case in which a shell suddenly enters a cross-wind region from a region of still air; it starts its relative trajectory with a yaw δ given by the equation

$$\tan \delta = w/v,$$

where w is the wind velocity and v the velocity of the shell. At the same time $\delta' = 0$ and $\phi' = 0$. The equations of § 3.6 enable the subsequent motion to be properly traced, and the errors in the usual treatment calculated.

§ 5.2. *Effect of Size and Shape of Shell.*

The jump card trials described in this paper were carried out with shells of two different shapes only. The differences between the two shells may be seen from fig. 6 to be considerable, form A having an ogival head of roughly 2 calibres radius, while form B is of 6 calibres radius. For form B the experiments determine the moment coefficient only, for a single position of the centre of gravity, and give no information as to the cross-wind force. As experiments of this type are expensive and laborious to carry out, it is of importance to examine how far these results may be applied to shells of other shapes and sizes.

From the results of § 1.1 it appears that there is no evidence that the size (represented by the radius r of the shell) enters into any of the factors on which the force coefficients depend, so that the coefficients f_R, f_M, f_L may be considered as entirely independent of size. It is therefore sufficient to make experiments on shells of as small a calibre as is consistent with obtaining accurate measurements of the jump cards.

With regard to the effect of variation of shape we have very little evidence.

If we compare the moment coefficients f_M for shells of 2 and 6 calibres radius of head, as shown in figs. 4 and 5, it is obvious that the difference is much less marked than the difference between the two curves of f_R, and that the two curves of f_M are very nearly of the same shape. No great errors would be introduced by assuming that the values of f_M for the two shells were in a constant ratio. Thus it seems reasonable, until the appearance of evidence to the contrary, to consider that the value of the moment coefficient for any shell can be obtained by multiplying the value, obtained in this experiment, by a constant independent of the velocity. It will then be sufficient to determine the value of this constant at a single velocity, which may even be a low velocity attainable in a wind channel. The value of the cross-wind force factor for any shell may be obtained in a similar manner but the results will be more uncertain. For rough purposes it may even be sufficient to assume f_L and f_M independent of the velocity except when dealing with velocities very near the velocity of sound. It thus appears to be possible to treat f_M in the classical way in which f_R was treated, in which it was assumed that the values of f_R for two different shells are in a constant ratio at all velocities. This treatment is inadequate in the case of f_R, but on present evidence is far more valid in the case of f_M.

By applying the results of the present trial in this way, we may even hope to get reasonably accurate estimates of the drift and stability for any type of shell, on the basis of wind channel experiments only on the particular shape of shell required. The method would be especially valuable in connection with the design of new shapes of shell. It is known that, in general, the longer and more pointed a shell is, the less is its drag coefficient; by a series of wind channel tests on a series of shell shapes it would be possible to determine the greatest length of shell that would be sufficiently stable in a gun of given rifling, or the sharpness of rifling required to make a given shell stable. Useful information was obtained on this point from wind channel experiments before the jump card trial provided certain data for the extrapolation to high velocities. It must be emphasised, however, that this one experiment needs extension and confirmation before the structure sketched above can confidently be reared upon it.

We have now discussed in general terms the applicability of our theory and experiments to the calculation of drift, stability, the effect of wind, and the design of improved forms of shell. Though the details of the calculations on these various points are not given here, enough has been said to show that the results form some advance in the subject of the application of aerodynamics to the flight of shells.

TABLE V.—General Table of Results.

Column 1. Number of round.
,, 2. Muzzle velocity, f.s., for round, or mean for group.
,, 3. Air density ρ, lb./(ft.)^3 and temperature ° F.
,, 4. $\phi' = \frac{1}{2}\Omega$, degrees/sec.
Upper entry—Calculated value for round, or mean for group.
Lower entry—Observed value.*
,, 5. Period T, between successive minima of δ, in units of 10^{-3} sec.
,, 6. Maxima of yaw, $\alpha_1(t)$ degrees.
,, 7. Mean values of ΩT, radians, for each round, for mean velocity as stated.
,, 8. Minima† of yaw, $\beta_1(t)$ degrees.

* When there is only one entry there was no detectable difference between the observed and calculated values of ϕ.

† *Note.*—The sign given is the sign of $\beta_1(t)$ at the minimum, see § 4.0. The yaw δ is always positive.

Gun rifled 1 turn in 40 calibres.

1.	2.	3.	4.	5.	6.	7.	8.
			Type I.				
I. 11	918	0·0792 43°	2108	252 267 —	2·1 2·5 2·1 ?	19·10 905	0·0 —
I. 12	918	0·0792 43°	2108	276 262 —	6·1 6·9 7·5 +	19·79 905	− 0·4 − 0·6
I. 14	920	0·0807 42°	2113	296 289	7·2 6·4	21·60 905	− 0·9 − 1·5 ?
I. 13	931	0·0807 42°	2139	234 260 —	1·5 3·1 5·5 +	18·44 919	− 1·0 − 2·4 ?
I. 5	1565	0·0782 45°	3595	230 —	12·2 8·5	28·87 1539	− 0·9
I. 6	1565	0·0782 45°	3595	227 —	13·7 9·5	28·49 1540	0·0

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
			Type I. (continued).				
I. 7	1565	0·0782 45°	3595 3375	354	10·5	41·71 1526	−2·3
I. 15	2130	0·0807 42°	4892	2T = 213	— ?7·5 5·5	18·19 2082	— −0·4
I. 16	2130	0·0807 42°	4892	2T = 203	— ?7·5 5·7	17·33 2085	— −0·6
I. 1	2167	0·0786 42°	4977	93½ 92 90½	4·5 4·1 3·7	15·98 2104	−1·0 −0·6 −0·6
I. 2	2167	0·0786 42°	4977	112 105 —	7·5 6·3 5·3	18·85 2117	0·0 −0·2 ?
I. 3	2167	0·0786 42°	4977	107 99½ —	4·5 3·5 3·1	17·94 2120	−0·3 −0·4
I. 4	2167	0·0786 42°	4977	118 121½	4·1 3·0	20·81 2113	−0·4 −1·4 ?
I. 19	2272	0·0812 40°	5217	98½ 90½ —	5·0 4·3 3·0	17·20 2217	−0·2 −0·3 ?
I. 20	2346	0·0812 40°	5388 5250	96½ 94½ —	8·7 7·9 7·1	17·96 2288	+0·4 +1·0
I. 21	2346	0·0812 40°	5388	99 99 —	5·6 4·0 3·3	18·03 2282	−1·0 −0·6

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
Type II. Form A. C.G. forward.							
II. 8	934	0·0807 42°	1960 2020	210 224 —	1·6 3·2 3·8	15·30 923	−0·9 −0·5
II. 9	934	0·0807 42°	1960	246 228 —	4·5 4·0 4·7	16·22 922	−0·3 −1·2
II. 10	934	0·0807 42°	1960	254 256 —	6·7 6·8 8·1	17·45 921	−0·8 −0·5
II. 5	1585	0·0780 46°	3541	147½ 145½ —	4·9 3·7 2·4	18·11 1554	−0·7 −0·7
II. 6	1585	0·0780 46°	3541	144½ 145½ —	1·2 1·2 1·0	17·92 1555	−0·5 ? −0·5 ?
II. 7	1585	0·0780 46°	3541	187 170	3·1 2·2	22·07 1548	−0·4 −0·3 ?
II. 1	2024	0·0786 42°	4795 4530	99 89½ 91½	3·2 2·5 2·3	14·76 1983	−0·7 −0·8 −0·7
II. 2	2024	0·0786 42°	4795 4625	103 95 87	3·0 2·3 1·9	15·34 1982	0·0 −0·1 −0·5
II. 3	2024	0·0786 42°	4795 4435	96 89 92	2·9 2·6 2·3	14·29 1984	−0·2 −0·2 −0·2
II. 4	2024	0·0786 42°	4795	98½ 81 85	1·9 1·7 1·8	14·75 1985	0·0 0·0 0·0

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
Type III. Form A. C.G. back.							
III. 8	931	0·0807 42°	2450	247 226 —	5·7 5·5 5·8	20·23 919	−0·4 −1·5
III. 9	931	0·0807 42°	2450	254 216 —	5·0 5·5 4·3	20·10 919	−0·4 −1·3
III. 10	931	0·0807 42°	2450	232 224 —	5·2 5·5 7·8	19·50 920	−0·3 −1·5
III. 5	1583	0·0780 46°	4166 4100	217 —	8·7 7·8	31·06 1556	−3·7
III. 6	1583	0·0780 46°	4166	196 —	9·0 8·2	28·52 1558	−3·6
III. 7	1583	0·0780 46°	4166 3980	237 —	5·5 6·3	32·93 1553	−2·9
III. 1	2025	0·0785 43°	5331	125 97 —	1·5 1·2 0·8	20·66 1994	— —
III. 2	2025	0·0785 43°	5331	113 97 —	3·7 3·4 3·4	19·54 1995	−0·7 −1·5
III. 3	2025	0·0785 43°	5331	109 107 —	2·4 1·8 1·3	20·10 1994	−0·2? −0·5
III. 4	2025	0·0785 43°	5331	106 102 —	2·9 2·1 1·8	19·36 1994	0·0 −0·8

TABLE V. (continued).

Gun rifled 1 turn in 30 calibres.

1.	2.	3.	4.	5.	6.	7.	8.
Type I. Form A. Empty shell.							
I. 22	1119	0·0811 38°	3429	91½ 86½ 82 86 79½ 87	2·1 1·9 1·9 — 1·9 2·0	10·07 1089	+0·1 0·0 −0·3 ? — −0·2 —
I. 23	1119	0·0811 38°	3429	79 91 87 86 83½ 84½	3·0 2·4 2·2 — 2·0 1·8	10·34 1090	+0·4 −0·1 −0·4 — −0·4 —
I. 24	1119	0·0811 38°	3429	85 90 86 84½ 84½ 84	2·8 2·4 2·2 — 2·1 2·1	10·27 1090	+0·4 −0·2 −0·4 — −0·6 −0·9 ?
I. 25	1326	0·0811 38°	4061	74½ 71 69 68½ 71 70 —	2·0 1·8 1·9 — 1·6 1·4 2·0	9·91 1280	0·0 0·0 0·0 — 0·0 −0·8 ?
I. 26	1326	0·0811 38°	4061	71 74 71 73 66½ 69½	3·6 2·8 2·6 — 2·6 2·2	10·04 1285	+0·4 +0·1 0·0 — −0·1 −0·8 ?
I. 27	1563	0·0805 36°	4786	60½ 55½ 59 60½ 55½ 56½ —	3·8 3·7 3·5 — 2·9 3·3 ? 2·5	9·59 1515	−0·3 −0·3 ? −0·2 — −0·5 −0·8

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
Type I. Form A. Empty shell (continued).							
I. 28	1563	0·0805	4786	59	3·1	9·66	−0·4
				59	2·6		−0·4
				59	2·1		−0·3
		36°		59	—		—
				57½	1·9		−0·2
				54½	2·0		−0·5 ?
				—	2·1	1515	
Type II. Form A. C.G. forward.							
II. 17	1119	0·0811	3168	92½	2·8	9·85	0·0
				90	2·75		0·0
				88½	2·6		−0·6
		38°	3128	91	—		−0·5
				91½	2·0		−0·7
				—	1·7	1092	
II. 18	1119	0·0811	3168	88	3·3	9·88	+0·1
				94½	2·8		0·0
				87½	2·6		−0·6
		38°	3128	89	—		−0·7
				91	2·3		−0·7
				—	2·2	1093	
II. 19	1119	0·0811	3168	88	2·9	9·82	+0·1
				89	2·4		0·0
				89½	2·0		−0·3
		38°	3128	90	—		−0·4
				89½	2·0		−0·5 ?
				91½	2·1	1088	−0·5 ?
II. 24	1292	0·0805	3709	76	2·7	9·70	−0·15
				74	2·9		−0·3
				72	2·7		−0·2
		36°		76	—		−0·3
				75	2·4		−0·7
				76	3·0	1259	−0·5
II. 22	1589	0·0819	4738	52	2·1	8·93	+0·5
				51	1·9		+0·1 ?
				54½	1·4		—
		36°		56½	1·4		—
				54½	—		+0·5
				52	1·1		—
				55½	1·4	1543	+0·1

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
Type II. Form A. C.G. forward (continued).							
II. 23	1589	0·0819 36°	4738	54 5 periods in 276	2·3 2·1 — — — 1·9 1·6	9·13 1548	+0·3 +0·1 — — — — −0·1
Type III. Form A. C.G. back.							
III. 17	1119	0·0811 38°	3928	67 75 71 73 71 74 77	1·8 1·8 2·2 2·2 — 1·8 1·7	10·08 1091	0·0 0·0 0·0 0·0 0·0 0·0 0·0
III. 18	1119	0·0811 38°	3928	63 71 75 77 72 71½ 67	1·4 1·4 1·4 ? 1·8 — 1·4 1·3	9·91 1091	+0·2 — +0·1 — +0·2 +0·1 −0·1
III. 19	1119	0·0811 38°	3928	64 72 77 74 74 71 77	1·6 1·4 1·3 1·4 — 1·1 1·1	10·17 1091	+0·1 +0·1 +0·1 +0·1 +0·1 −0·1 —
III. 20	1292	0·0805 36°	4534	62 59 62 58 63 63 60 —	2·7 2·7 2·2 2·2 — 2·0 1·6 1·6	9·63 1261	0·0 −0·1 −0·5 — −1·0 −0·5 −0·9

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
Type III. Form A. C.G. back (continued).							
III. 21	1292	0·0805 36°	4534	60 62 59 59 61 59 60	3·6 3·2 2·7 2·6 — 2·3 2·1	9·50 1262	0·0 0·0 −0·2 ? — −0·7 ? −0·5 −0·6
III. 22	1567	0·0805 36°	5501	42 50 49 49 50 49 46 —	3·5 2·9 2·8 2·4 — 2·6 — 2·3	9·38 1525	−0·1 −0·2 −0·4 — −0·4 −0·3 −0·3
III. 23	1567	0·0805 36°	5501	51 ? 42 51 47 48 ? 51 ? 49 ?	1·7 ? 1·6 1·4 1·0 — 1·0 ? — 1·0	9·22 1526	0·0 0·0 0·0 — — 0·0 0·0 0·0
Type IV. Form B.							
IV. 21	900	0·0811 38°	2431	116 120 122 116 126 —	1·7 2·8 2·3 — 2·7 + 3·5 +	10·27 884	−0·6 −1·5 — −1·0 −2·0 ?
IV. 22	900	0·0811 38°	2431	124 128 119 110 119 —	3·0 2·8 1·9 — 2·3 3·0 +	10·10 884	+0·3 −0·2 — −0·4 −0·7

TABLE V. (continued).

Type IV. Form B (continued).

1.	2.	3.	4.	5.	6.	7.	8.
IV. 23	900	0·0811 38°	2431	— 112 120 114 118 —	1·7 ? 1·7 2·6 — 2·0 2·0+	9·84 884	+0·2 −0·8 — −0·8 −1·0
IV. 13	1078	0·0811 38°	2911	113 128 125 113 —	2·0 1·8 1·5 ? 1·6 2·0	12·39 1059	+0·3 −0·4 −0·6 −0·9
IV. 14	1078	0·0811 38°	2911	109 123 110 105 —	1·6 1·5 1·8 ? 1·6 1·7	11·45 1060	+0·2 −0·3 — −0·3
IV. 15	1078	0·0811 38°	2911	107 121 118 108 —	2·2 2·1 2·2 1·7 1·8	11·76 1060	+0·3 −0·1 — −0·5
IV. 16	1547	0·0811 38°	4178	82 78 67 75 74 ?	1·6 1·6 1·4 1·0 0·9	10·72 1503	+0·1 0·0 — 0·0 −0·1
IV. 17	1547	0·0811 38°	4178	83 75 75 73 71	3·5 3·2 2·8 1·9 2·6	10·72 1503	+0·1 0·0 ? — −1·2 ? −0·9 ?
IV. 18	1547	0·0811 38°	4178	83 80 75 73 —	4·2 3·9 3·3 3·2 3·1	11·09 1509	+0·1 0·0 — −0·4 −1·1

TABLE V. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
			Type IV. Form B (continued).				
IV. 19	1547	0·0811 38°	4178	77 75 73	4·0 3·4 3·4	10·96 1496	—
IV. 20	1547	0·0811 38°	4178	73 75 72	3·3 2·5 2·2	10·69 1497	—
IV. 24	2101	0·0805 36°	5675	55 57 51 55 53	4·6 3·8 3·5 3·5 3·4	10·70 2045	−0·2 −0·3 — −0·5 ? −1·0 ?
IV. 25	2112	0·0805 36°	5705	56 55 52 55 54	2·2 2·2 1·3 1·2 1·3	10·70 2073	0·0 ? 0·0 — 0·0 ? −0·8
IV. 26	2149	0·0805 36°	5805	50 53 54 53 54	2·4 2·0 2·0 1·7 1·2	10·60 2093	−0·1 0·0 — — −0·3

TABLE VI.—Values of the Stability Coefficient and the Air Couple deduced from Analysis of the *Stable* Shells.

Summary of Notation used in the Headings of this Table.

$\mu \sin \delta$ = couple due to air forces.

$s = \frac{\Omega^2 B}{4\mu}$ = stability coefficient.

v = mean velocity of shell, f.s.

ρ = air density, lb./(ft.)^3.

a = velocity of sound, f.s.

$f_M(v/a) = \mu/(\rho v^2 r^3)$, the air couple coefficient.

N.B.—The values of s, μ and f_M have not been corrected in this table for the effect of the cards.

1.	2.	3.	4.	5.	6.	7.	8.
Round No. or group of rounds.	Twist of rifling.	Value of s deduced from observation.	Mean value of v corresponding to value of s.	Value of μ.	Value of $f_M(v/a)$.	Value of v/a.	Total percentage spread of s or μ in group.
I. 11, 12	1/40	1·113	905	1220	9·58	0·824	4·1
I. 14	1/40	1·087	906	1250	9·71	0·825	
I. 13	1/40	1·131	919	1230	9·26	0·837	
I. 22–24	1/30	1·61	1090	2230	11·85	0·996	3·2
I. 25, 26	1/30	1·66	1283	3030	11·62	1·173	1·7
I. 27, 28	1/30	1·74	1515	4000	11·09	1·388	1·1
I. 5– 7	1/40	1·005	1535	3920	10·89	1·394	0·9
I. 15, 16	1/40	1·137	2084	6410	9·36	1·897	6·7
I. 1	1/40	1·180	2104	6390	9·40	1·916	
I. 2– 4	1/40	1·118	2117	6750	9·80	1·927	
I. 19	1/40	1·152	2217	7190	9·22	2·023	1·9
I. 20, 21	1/40	1·133	2285	7800	9·41	2·085	
II. 8–10	1/40	1·172	922	1170	8·71	0·840	4·9
II. 17–19	1/30	1·69	1091	2000	10·46	0·997	0·8
II. 24	1/30	1·72	1259	2700	10·86	1·153	
II. 5– 7	1/40	1·121	1553	3680	9·98	1·408	4·7
II. 22, 23	1/30	1·94	1546	3800	9·95	1·416	3·8

TABLE VI. (continued).

1.	2.	3.	4.	5.	6.	7.	8.
Round No. or group of rounds.	Twist of rifling.	Value of s deduced from observation.	Mean value of v corresponding to value of s.	Value of μ.	Value of $f_M(v/a)$.	Value of v/a.	Total percentage spread of s or μ in group.
II. 1	1/40	1·216	1983	5506	9·12	1·804	
II. 2	1/40	1·200	1982	5689	9·42	1·803	
II. 3	1/40	1·232	1984	5320	8·80	1·805	
II. 4	1/40	1·220	1985	5809	9·59	1·806	
III. 8–10	1/40	1·107	919	1520	11·41	0·837	0·9
III. 17–19	1/30	1·64	1091	2570	13·66	0·997	3·3
III. 20, 21	1/30	1·76	1262	3200	12·79	1·156	2·1
III. 22, 23	1/30	1·84	1526	4500	12·32	1·398	2·9
III. 5– 7	1/40	1·035	1556	4410	11·96	1·411	0·6
III. 1– 4	1/40	1·109	1994	7020	11·52	1·814	1·4
IV. 21–23	1/30	1·64	884	1270	10·25	0·808	2·6
IV. 13–15	1/30	1·39	1060	2140	12·06	0·969	5·0
IV. 16–20	1/30	1·505	1502	4070	11·41	1·373	3·7
IV. 24	1/30	1·525	2045	7420	11·29	1·874	
IV. 25	1/30	1·525	2075	7500	11·11	1·899	1·0
IV. 26	1/30	1·54	2093	7680	11·16	1·917	

TABLE VII.—Observed Values of $h+\kappa$ and $h-\kappa+2\gamma$, the Damping Factors for each Group.

Groups fired at a velocity of 900 f.s. apparently have negative damping and are not included.

Group. Muzzle velocity, f.s.	Calculated value of κ.	$h+\kappa$.	$h-\kappa+2\gamma$.	Probable value of h.
I. 22–24 1119	0·4	1·9	1·6 ?	1·8
I. 25, 26 1326	0·3	2·4	1·2	1·8
I. 27, 28 1563	0·4	3·0	0·5	1·8
I. 1– 4 2167	0·7	2·2	0·6	1·4
I. 19–21 2320	0·8	2·2	−0·2 ?	1·3
II. 17–19 1119	0·4	2·2	1·2	1·7
II. 24 1292	0·2	0·9	0·6	1·5
II. 5– 7 1585	0·4	3·4	0·4	2·0
II. 22, 23 1589	0·4	3·3	0·6	
II. 1– 4 2027	0·6	3·0	0·6	2·0
III. 17–19 1119	0·4	0·7 ?	0·1 ?	1·0
III. 20, 21 1292	0·2	3·1	0·9	2·0
III. 22, 23 1567	0·4	3·0	0·3	2·0
III. 1– 4 2025	0·6	4·2	1·7	3·0
IV. 13–15 1078	0·5	0·7	1·4	1·0
IV. 16–18 1547	0·5	3·1	1·2	2·0
IV. 24–26 2120	0·7	5·0	0·9	3·0

N.B.—The calculated value of κ is obtained by using the value of the cross-wind force coefficient given in Table I.

TABLE VIII.—Plane Trajectories at 50 degrees and 30 degrees, with Calculations of the Drift, &c., for Shells of External Form A.

Constants used in the Calculations.

Muzzle velocity	2000 feet per second.
Centre of gravity	4·88 inches from the base.
Weight	16·02 lb.
Moments of inertia A	0·1329 lb. (ft.)^2.
Moments of inertia B	1·1555 lb. (ft.)^2.
Ω for gun rifled 1 in 30 . . .	192·7 radians per second.
Ω for gun rifled 1 ,, 40 . . .	144·5 ,, ,,
Ballistic coefficient.	1·75.

TABLE VIIIA.—Plane Trajectories at Elevations of 50 degrees and 30 degrees.

Column 1. Time t, seconds.
,, 2. Velocity v_1, feet per second.
,, 3. Inclination θ_1, degrees.
,, 4. Horizontal distance X, feet.
,, 5. Vertical height Y, feet.

	Elevation 50 degrees.				Elevation 30 degrees.			
1.	2.	3.	4.	5.	2.	3.	4.	5.
		° ′				° ′		
0	2000	50 0	0	0	2000	30 0	0	0
1	1720	49 21	1,199	1,413	1726	29 8	1,614	917
2	1506	48 36	2,254	2,628	1515	28 8	3,033	1692
3	1342	47 44	3,201	3,686	1352	26 59	4,300	2354
4	1218	46 45	4,068	4,624	1230	25 42	5,454	2926
6	1059	44 28	5,648	6,241	1075	22 45	7,538	3867
8	959	41 48	7,116	7,619	985	19 24	9,455	4608
10	877	38 43	8,514	8,805	916	15 42	11,264	5182
12	807	35 13	9,857	9,819	860	11 40	12,988	5603
14	746	31 15	11,155	10,671	816	7 19	14,639	5881
16	693	26 45	12,411	11,369	780	2 43	16,227	6021
18	647	21 44	13,631	11,921	751	− 2 7	17,755	6030
20	609	16 10	14,817	12,330	728	− 7 5	19,228	5912
24	557	3 40	17,098	12,738	703	−17 11	22,017	5314
28	537	− 9 50	19,266	12,624	700	−26 55	24,608	4261
32	547	−22 55	21,331	12,013	712	−35 51	27,011	2790
36	579	−34 25	23,293	10,930	735	−43 40	29,228	938
40	627	−43 54	25,151	9,403				
44	681	−51 30	26,902	7,466				
48	735	−57 32	28,539	5,157				
52	786	−62 23	30,057	2,521				
55	818	−65 25	31,114	359				

TABLE VIIIB.—Calculation of the Drift, Stability, and Damping Factors, for the Gun Rifled 1 in 30.

Column 1. The time t, seconds.
,, 2. The stability factor s.
,, 3. $-4s\theta'_1/\Omega$.
,, 4. $4s\kappa/\Omega$.
,, 5. ψ, equation (4.203).
,, 6. The drift Z, feet.
,, 7. The azimuth, arc tan (Z/X), degrees.
,, 8. The second term in the expansion of $\bar{\eta}$, equation (3.632), given by

$$\frac{\eta^{(2)}}{(i\Omega)^2} = \frac{4s}{(i\Omega)^2}\left\{\frac{d}{dt}(4s\theta'_1) + 4s\kappa\theta'_1 + c'\int_0^t 4s\kappa\theta'_1\,dt/c\right\}.$$

,, 9. The damping factor (§ 4.22),

$$(\sigma_0/\sigma)^{\frac{1}{2}}\,e^{-(q_1-q_2)}.$$

Elevation 50 degrees.

1.	2.	3.	4.	5.	6.	7.	8.	9.
						°		
0	1·945	0·00042	0·0217	0	0			1
1	2·598	0·00062		0·00038	0·2			0·702
2	3·334	0·00098	0·0214	0·00082	0·8			0·520
3	4·189	0·00139		0·00129	1·8			0·404
4	5·129	0·00192	0·0221	0·00182	3·1		0·00007	0·330
6	7·123	0·00321		0·00405	7·5	0 5		0·196
8	10·95	0·00569	0·0570	0·00752	15·9		0·00066	0·115
10	14·60	0·00868		0·0114	29·1	0 12		0·076
12	18·39	0·01240	0·0518	0·0155	47·3	0 16	0·00155	0·055
14	22·20	0·01693		0·0199	70·4			0·043
16	26·33	0·0227	0·0585	0·0250	98·6	0 27	0·00334	0·034
18	30·88	0·0297		0·0308	133			
20	35·16	0·0372	0·0663	0·0374	173		0·0056	
24	42·63	0·0511		0·0530	276	0 55		
28	45·87	0·0562	0·0755	0·0705	409	1 13		
32	43·24	0·0486		0·0885	573	1 32		
36	37·11	0·0353	0·0700	0·1050	762		0·0012	
40	30·11	0·0231		0·1192	971	2 12		
44	24·02	0·0147	0·0595	0·1315	1190			
48				0·1420	1415		−0·0019	
52				0·1510	1637			

TABLE VIIIB. (continued).

Elevation 30 degrees.

1.	2.	3.	4.	5.	6.	7.
						° ′
0	1·945	0·00056		0	0	
1	2·55	0·00086				
2	3·21	0·00122		0·00080	1·1	
4	4·79	0·00234		0·00174	4·1	
6	8·43	0·00368		0·00351	9·5	
8	9·11	0·00581		0·00655	19·2	0 7
10	11·16	0·00785		0·01022	34·4	
12	14·01	0·01063		0·01417	55·3	0 15
14	15·75	0·01276		0·01828	82·0	
16	17·48	0·01495		0·0225	114·3	0 24
18	18·85	0·01671		0·0269	152·0	
20	19·92	0·01810		0·0316	195·0	0 35
24		0·0191		0·0418	297	0 46
28		0·0174		0·0525	419	0 59
32		0·0143		0·0629	557	1 11
36		0·0110		0·0729	708	1 23

TABLE IX.—Comparison of Calculated Drift with Observations of April–May and February, 1918.

The azimuth of the shell at time t (in minutes of angle) = At.

Elevation 50 degrees.

Observations of April–May.

Rifling 1/30.			Rifling 1/40.		
Mean observed time.	Mean observed A.	Calculated A.	Mean observed time.	Mean observed A.	Calculated A.
10·9	1·46	1·27	10·2	1·18	0·90
23·9	2·24	2·29	22·9	1·30	1·66
33·3	2·89	2·85	31·0	1·95	2·10
41·3	3·16	3·36	39·1	2·00	2·44

TABLE IX. (continued).

Elevation 50 degrees (continued).

Observations of February.

Rifling 1/30.			Rifling 1/40.		
Mean observed time.	Mean observed A.	Calculated A.	Mean observed time.	Mean observed A.	Calculated A.
6·99	4·54	0·86	6·33	3·61	0·63
15·03	1·68	1·65	14·07	1·40	1·18
26·08	1·99	2·46	24·93	1·47	1·86

Elevation 30 degrees.

Observations of April–May.

Rifling 1/30.			Rifling 1/40.		
Mean observed time.	Mean observed A.	Calculated A.	Mean observed time.	Mean observed A.	Calculated A.
10·04	0·975	1·05	9·58	1·63	0·79
20·6	1·575	1·77	19·35	0·80	1·28
27·9	2·09	2·06	25·95	1·04	1·50

Observations of February.

Rifling 1/30.			Rifling 1/40.		
Mean observed time.	Mean observed A.	Calculated A.	Mean observed time.	Mean observed A.	Calculated A.
13·2	1·40	1·32	13·02	1·73	0·97
22·52	1·36	1·86	22·05	1·25	1·38

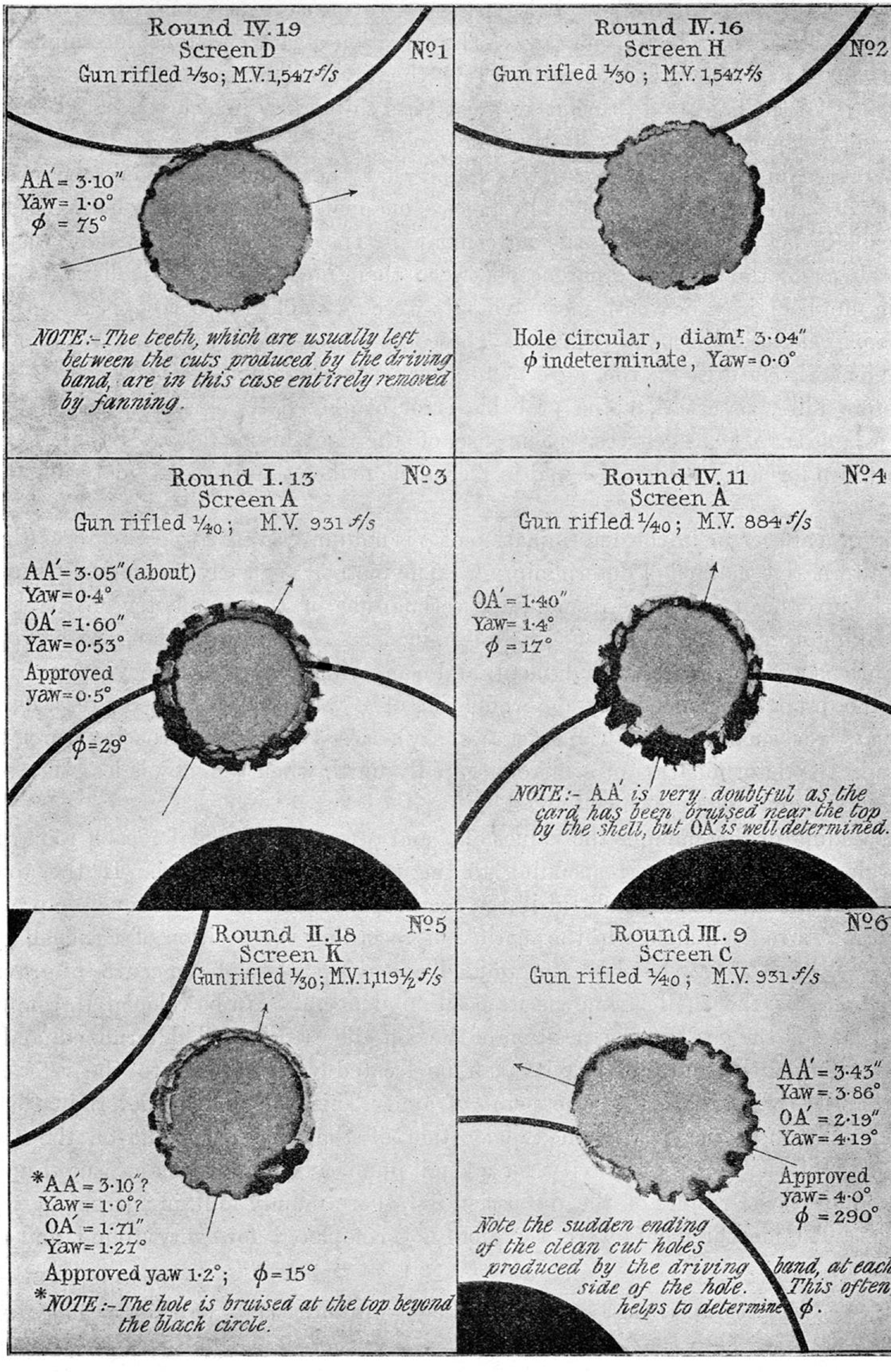

Round IV. 19
Screen D
Nº 1
Gun rifled 1/30; M.V. 1,547 f/s
AA′ = 3·10″
Yaw = 1·0°
φ = 75°
NOTE:- The teeth, which are usually left between the cuts produced by the driving band, are in this case entirely removed by fanning.
Round IV. 16
Screen H
Nº 2
Gun rifled 1/30; M.V. 1,547 f/s
Hole circular, diamr. 3·04″
φ indeterminate, Yaw = 0·0°
Round I. 13
Nº 3
Screen A
Gun rifled 1/40; M.V. 931 f/s
AA′ = 3·05″ (about)
Yaw = 0·4°
OA′ = 1·60″
Yaw = 0·53°
Approved yaw = 0·5°
φ = 29°
Round IV. 11
Nº 4
Screen A
Gun rifled 1/40; M.V. 884 f/s
OA′ = 1·40″
Yaw = 1·4°
φ = 17°
NOTE:- AA′ is very doubtful as the card has been bruised near the top by the shell, but OA′ is well determined.
Round II. 18
Nº 5
Screen K
Gun rifled 1/30; M.V. 1,119½ f/s
*AA′ = 3·10″?
Yaw = 1·0°?
OA′ = 1·71″
Yaw = 1·27°
Approved yaw 1·2°; φ = 15°
*NOTE:- The hole is bruised at the top beyond the black circle.
Round III. 9
Nº 6
Screen C
Gun rifled 1/40; M.V. 931 f/s
AA′ = 3·43″
Yaw = 3·86°
OA′ = 2·19″
Yaw = 4·19°
Approved yaw = 4·0°
φ = 290°
Note the sudden ending of the clean cut holes produced by the driving band, at each side of the hole. This often helps to determine φ.

Fig. 8A.